

# Brauer algebras of type B

Arjeh M. Cohen, Shoumin Liu

December 22, 2011

## Abstract

For each  $n \geq 1$ , we define an algebra having many properties that one might expect to hold for a Brauer algebra of type  $B_n$ . It is defined by means of a presentation by generators and relations. We show that this algebra is a subalgebra of the Brauer algebra of type  $D_{n+1}$  and point out a cellular structure in it. This work is a natural sequel to the introduction of Brauer algebras of type  $C_n$ , which are subalgebras of classical Brauer algebras of type  $A_{2n-1}$  and differ from the current ones for  $n > 2$ .

## 1 Introduction

In [4], the Brauer algebra  $\text{Br}(Q)$  of any simply-laced Coxeter type was defined in such a way that for  $Q = A_{n-1}$ , the classical Brauer algebra of diagrams on  $2n$  nodes emerges. For these algebras, a deformation to a Birman–Murakami–Wenzl (BMW) algebra was defined and in [5], and, for the spherical types among these, the algebra structure was fully determined in [6, 9]. Again, for  $Q = A_{n-1}$ , the classical BMW algebras re-appear.

The starting point for an extension of these algebras to non-simply laced diagrams, begun in [8], is based on the following observation. It is well known that the Coxeter group of type  $B_n$  arises from the Coxeter group of type  $D_{n+1}$  as the subgroup of all elements fixed by the nontrivial Coxeter diagram automorphism. Crisp [10] showed that the Artin group of type  $B_n$  arises in a similar fashion from the Artin group of type  $D_{n+1}$ . In this paper, we study the subalgebra  $\text{SBr}(D_{n+1})$  of the Brauer algebra  $\text{Br}(D_{n+1})$  spanned by the monomials fixed under the automorphism induced by the nontrivial Coxeter diagram automorphism. We also give a presentation of this subalgebra by generators and relations, which we regard as the definition of a Brauer algebra of type  $B_n$ . This paper continues the introduction in [8] of a Brauer algebra of type  $C_n$  of  $\text{Br}(A_{2n-1})$  spanned by monomials fixed under the canonical Coxeter diagram automorphism.

Each defining relation (given in Definition 2.1 below) concerns at most two indices, say  $i$  and  $j$ , and is (up to the parameters in the idempotent relation) determined by the diagram induced by  $B_n$  on  $\{i, j\}$ . The nodes of the Dynkin type  $B_n$  are labeled as follows.

$$B_n = \begin{array}{c} \circ \text{---} \circ \cdots \cdots \circ \text{---} \circ \Rightarrow \circ \\ n-1 \quad n-2 \quad \quad \quad 2 \quad 1 \quad \quad \quad 0 \end{array}$$

The generators of the Brauer algebra  $\text{Br}(B_n)$  are denoted  $r_0, \dots, r_{n-1}, e_0, \dots, e_{n-1}$ . In order to distinguish these from the canonical generators of the Brauer algebra of type  $D_{n+1}$ , the latter are denoted  $R_1, \dots, R_{n+1}, E_1, \dots, E_{n+1}$  instead of the usual lower case letters (see Definition 2.3). The diagram for  $D_{n+1}$  is depicted below.

$$D_{n+1} = \begin{array}{ccccccc} & & & & & 2 & \\ & & & & & \circ & \\ & & & & & | & \\ n_{+1} & - & \circ & - & \circ & \cdots & \cdots & \circ & - & \circ & - & \circ & - & \circ & - & \circ \\ & & n_+ & & 4 & & 3 & & 1 \end{array}$$

**Definition 1.1.** Let  $\sigma$  be the natural isomorphism on  $\text{BrM}(\mathbf{D}_{n+1})$  (Definition 2.3), which is induced by the action of the permutation  $(1, 2)$  on the indices of the generators  $\{R_i, E_i\}_{i=1}^{n+1}$  of  $\text{BrM}(\mathbf{D}_{n+1})$  and keeping the parameter  $\delta$  invariant. The fixed submonoid of  $\text{BrM}(\mathbf{D}_{n+1})$  under  $\sigma$  is called the *symmetric submonoid* of  $\text{BrM}(\mathbf{D}_{n+1})$ , and denoted  $\text{SBrM}(\mathbf{D}_{n+1})$ . The linear span of  $\text{SBrM}(\mathbf{D}_{n+1})$  is called the *symmetric subalgebra* of  $\text{Br}(\mathbf{D}_{n+1})$ , and denoted  $\text{SBr}(\mathbf{D}_{n+1})$ .

**Theorem 1.2.** *There exists a  $\mathbb{Z}[\delta^{\pm 1}]$ -algebra isomorphism*

$$\phi : \mathrm{Br}(\mathbf{B}_n) \longrightarrow \mathrm{SBr}(\mathbf{D}_{n+1})$$

determined by  $\phi(r_0) = R_1 R_2$ ,  $\phi(r_i) = R_{i+2}$ ,  $\phi(e_0) = E_1 E_2$ , and  $\phi(e_i) = E_{i+2}$ , for  $0 < i \leq n-1$ . Furthermore both algebras are free of rank

$$f(n) := 2^{n+1} \cdot n!! - 2^n \cdot n! + (n+1)!! - (n+1)!$$

In Theorem 5.1 of this paper, we also show that these algebras are cellular in the sense of Graham and Lehrer [14]. The subalgebra of  $\text{Br}(\mathbf{B}_n)$  generated by  $r_0, \dots, r_{n-1}$  is easily seen to be isomorphic to the group algebra of the Weyl group  $W(\mathbf{B}_n)$  of type  $\mathbf{B}_n$ .

Here we give the first few values of ranks of  $\text{Br}(\mathbb{B}_n)$ .

$n$	1	2	3	4	5
$f(n)$	3	25	273	3801	66315

The case  $n = 2$  is discussed in [8], as  $B_2$  and  $C_2$  represent the same diagram. Here we illustrate our results with the next interesting case:  $n = 3$ . Let

$$F = \{1, e_2, e_0, e_1 e_0, e_0 e_1, e_1 e_0 e_1, e_0 e_2, e_2 r_1 e_0 e_1 e_2\}.$$

We have the following decomposition of  $\text{Br}(B_3)$  into  $\mathbb{Z}[\delta^{\pm 1}]W(B_3)$ -submodules, where  $W(B_3)$  is the submonoid of  $\text{Br}(B_3)$  generated by  $r_0, r_1, r_2$ .

$$\text{Br}(B_3) = \bigoplus_{e \in F} \mathbb{Z}[\delta^{\pm 1}]W(B_3)eW(B_3).$$

Besides, the sizes of  $W(B_3)eW(B_3)$  are 48, 144, 18, 18, 18, 9, 9, 9 for  $e \in F$  in the order they are listed. This accounts for the rank of  $\text{Br}(B_3)$  being 273.

The strategy of proof is as follows. The monomials in the canonical generators of  $\text{Br}(D_{n+1})$  are known to correspond to certain Brauer diagrams with an additional decoration of order two by means of the isomorphism  $\psi : \text{Br}(D_{n+1}) \rightarrow \text{BrD}(D_{n+1})$  introduced in [7]; here  $\text{BrD}(D_{n+1})$  is an algebra linearly spanned by the decorated classical Brauer diagrams, and the details are described in Section 2. The proof then consists of showing that the image of  $\phi$  is the linear span of the symmetric diagrams, which is free of rank  $f(n)$ , and that  $\text{Br}(B_n)$  is linearly spanned by at most  $f(n)$  monomials. The latter is carried out by means of rewriting monomials to normal forms, in such a way that each Brauer diagram corresponds to a unique normal form. This process leads to a basis which can be shown to be cellular.

This paper has five sections. Section 2 gives the definition of  $\text{Br}(B_n)$  and some elementary properties of  $\text{Br}(B_n)$  in preparation of Section 4. We also recall results of Brauer algebras of type  $D_{n+1}$  and present the surjectivity of the map  $\phi$  of Theorem 2.11 by combinatorial arguments in this section. Section 3 discusses aspects of the root system of type  $B_n$  that are used to identify monomials of  $\text{Br}(B_n)$  in  $r_0, \dots, r_{n-1}, e_0, \dots, e_{n-1}$ ; moreover a pictorial description of some monomial images in  $\text{BrD}(D_{n+1})$  is presented. In Section 4, the rewriting of monomials of  $\text{Br}(B_n)$  is established, which leads to an upper bound on the rank of  $\text{Br}(B_n)$  and the main theorem is proved at the end of this section. In the last section, we establish that  $\text{Br}(B_n)$  is cellular.

The idea of obtaining non-simply laced algebras from simply laced types has been applied in [11] for Temperley-Lieb algebras of type B with generators  $e_i$ , in [16] for the reduced BMW algebras of type B and in [13] for Hecke algebras of type B. In [3], Z. Chen defines a Brauer algebra for each pseudo-reflection group by use of a flat connection. For types B and C, it is different from our algebra and has some intricate relations with our algebras to be explained in further research.

## 2 Definition and elementary properties

All our rings and algebras here will be unital (i.e., have an identity element) and associative.

**Definition 2.1.** Let  $\mathbb{Z}[\delta^{\pm 1}]$  be the group ring over  $\mathbb{Z}$  of the infinite cyclic group with generator  $\delta$ . For  $n \in \mathbb{N}$ , the *Brauer algebra of type  $B_n$  over  $\mathbb{Z}[\delta^{\pm 1}]$* , denoted by  $\text{Br}(B_n)$ , is the  $\mathbb{Z}[\delta^{\pm 1}]$ -algebra generated by  $r_0, r_1, \dots, r_{n-1}$  and  $e_0, e_1, \dots, e_{n-1}$  subject to the following relations.

$$r_i^2 = 1 \quad \text{for any } i \quad (2.1)$$

$$r_i e_i = e_i r_i = e_i \quad \text{for any } i \quad (2.2)$$

$$e_i^2 = \delta e_i \quad \text{for } i > 0 \quad (2.3)$$

$$e_0^2 = \delta^2 e_0 \quad (2.4)$$

$$r_i r_j = r_j r_i \quad \text{for } i \perp j \quad (2.5)$$

$$e_i r_j = r_j e_i \quad \text{for } i \perp j \quad (2.6)$$

$$e_i e_j = e_j e_i \quad \text{for } i \perp j \quad (2.7)$$

$$r_i r_j r_i = r_j r_i r_j \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.8)$$

$$r_j r_i e_j = e_i e_j \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.9)$$

$$r_i e_j r_i = r_j e_i r_j \quad \text{for } i \sim j \text{ with } i, j > 0 \quad (2.10)$$

$$r_1 r_0 r_1 r_0 = r_0 r_1 r_0 r_1 \quad (2.11)$$

$$r_0 r_1 e_0 = r_1 e_0 \quad (2.12)$$

$$r_0 e_1 r_0 e_1 = e_1 e_0 e_1 \quad (2.13)$$

$$(r_0 r_1 r_0) e_1 = e_1 (r_0 r_1 r_0) \quad (2.14)$$

$$e_0 r_1 e_0 = \delta e_0 \quad (2.15)$$

$$e_0 e_1 e_0 = \delta e_0 \quad (2.16)$$

$$e_0 r_1 r_0 = e_0 r_1 \quad (2.17)$$

$$e_0 e_1 r_0 = e_0 e_1 \quad (2.18)$$

Here  $i \sim j$  means that  $i$  and  $j$  are adjacent in the Dynkin diagram  $B_n$ , and  $\perp$  indicates that they are distinct and non-adjacent. The submonoid of the multiplicative monoid of  $\text{Br}(B_n)$  generated by  $\delta, \delta^{-1}, \{r_i\}_{i=0}^{n-1}$ , and  $\{e_i\}_{i=0}^{n-1}$  is denoted by  $\text{BrM}(B_n)$ . It is the monoid of monomials in  $\text{Br}(B_n)$  and will be called *the Brauer monoid of type  $B_n$* .

It is a direct consequence of the definition that the submonoid of  $\text{BrM}(B_n)$  generated by  $\{r_i\}_{i=0}^{n-1}$  (the Weyl group generators) is isomorphic to the Weyl group  $W(B_n)$  of type  $B_n$ . The algebra  $\text{Br}(B_2)$  is isomorphic to  $\text{Br}(C_2)$  defined in [8], and the isomorphism is given by exchanging the indices 0 and

1 of the Weyl group generators and of the Temperley-Lieb generators. As a consequence, Lemma 4.1 of [8] applies in the following sense.

**Lemma 2.2.** *In  $\text{Br}(\text{B}_n)$ , the following equalities hold.*

$$e_1 e_0 e_1 = e_1 r_0 e_1 \quad (2.19)$$

$$r_0 e_1 e_0 = e_1 e_0 \quad (2.20)$$

$$e_0 r_1 r_0 e_1 = e_0 e_1 \quad (2.21)$$

$$r_1 r_0 e_1 r_0 = r_0 e_1 r_0 r_1 \quad (2.22)$$

$$e_1 r_0 e_1 r_0 = e_1 e_0 e_1 \quad (2.23)$$

We recall from [4] the definition of a Brauer algebra of simply laced Coxeter type  $Q$ . In order to avoid confusion with the above generators, the symbols of [4] for the generators of  $\text{Br}(Q)$  have been capitalized.

**Definition 2.3.** Let  $Q$  be a simply laced Coxeter graph. The *Brauer algebra of type  $Q$  over  $R$  with loop parameter  $\delta$* , denoted  $\text{Br}(Q)$ , is the algebra over  $\mathbb{Z}[\delta^{\pm 1}]$  generated by  $R_i$  and  $E_i$ , for each node  $i$  of  $Q$  subject to the following relations, where  $\sim$  denotes adjacency between nodes of  $Q$  and  $\perp$  non-adjacency for distinct nodes.

$$R_i^2 = 1 \quad (2.24)$$

$$E_i^2 = \delta E_i \quad (2.25)$$

$$R_i E_i = E_i R_i = E_i \quad (2.26)$$

$$R_i R_j = R_j R_i \quad \text{for } i \perp j \quad (2.27)$$

$$E_i R_j = R_j E_i \quad \text{for } i \perp j \quad (2.28)$$

$$E_i E_j = E_j E_i \quad \text{for } i \perp j \quad (2.29)$$

$$R_i R_j R_i = R_j R_i R_j \quad \text{for } i \sim j \quad (2.30)$$

$$R_j R_i E_j = E_i E_j \quad \text{for } i \sim j \quad (2.31)$$

$$R_i E_j R_i = R_j E_i R_j \quad \text{for } i \sim j \quad (2.32)$$

As before, we call  $\text{Br}(Q)$  the *Brauer algebra of type  $Q$*  and denote by  $\text{BrM}(Q)$  the submonoid of the multiplicative monoid of  $\text{Br}(Q)$  generated by all  $R_i$  and  $E_i$ ,  $\delta$ , and  $\delta^{-1}$ .

For each  $Q$ , the algebra  $\text{Br}(Q)$  is free over  $\mathbb{Z}[\delta^{\pm 1}]$ . The classical Brauer algebra on  $m + 1$  strands arises when  $Q = A_m$ .

*Remark 2.4.* It is straightforward to show that the following relations hold in  $\text{Br}(Q)$  for all nodes  $i, j, k$  with  $i \sim j \sim k$  and  $i \perp k$ .

$$E_i R_j R_i = E_i E_j \quad (2.33)$$

$$R_j E_i E_j = R_i E_j \quad (2.34)$$

$$E_i R_j E_i = E_i \quad (2.35)$$

$$E_j E_i R_j = E_j R_i \quad (2.36)$$

$$E_i E_j E_i = E_i \quad (2.37)$$

$$E_j E_i R_k E_j = E_j R_i E_k E_j \quad (2.38)$$

$$E_j R_i R_k E_j = E_j E_i E_k E_j \quad (2.39)$$

As in [8, Remark 3.5], there is a natural anti-involution on  $\text{Br}(\mathbb{B}_n)$ . This anti-involution is denoted by the superscript  $_{\text{op}}$ , so the map is denoted by  $x \mapsto x^{\text{op}}$  for any  $x \in \text{Br}(\mathbb{B}_n)$ .

**Proposition 2.5.** *The identity map on  $\{\delta, r_i, e_i \mid i = 0, \dots, n-1\}$  extends to the anti-involution  $x \mapsto x^{\text{op}}$  on the Brauer algebra  $\text{Br}(\mathbb{B}_n)$ .*

Since  $\text{Br}(\mathbb{B}_2) \cong \text{Br}(\mathbb{C}_2)$  and  $\text{Br}(\mathbb{D}_3) \cong \text{Br}(\mathbb{A}_3)$ , the following corollary can be verified easily as in [8].

**Corollary 2.6.** *The map defined as  $\phi$  on the generators of  $\text{Br}(\mathbb{B}_n)$  in Theorem 1.2 extends to a unique algebra homomorphism  $\phi$  on  $\text{Br}(\mathbb{B}_n)$ . Furthermore, the image of  $\phi$  is contained in  $\text{SBr}(\mathbb{D}_{n+1})$ .*

*Proof.* The first claim can be verified by checking defining relations under  $\phi$ . The second claim holds for the image of each generator of  $\text{Br}(\mathbb{B}_n)$  under  $\phi$  is in  $\text{SBr}(\mathbb{D}_{n+1})$ .  $\square$

Since the subalgebra in  $\text{Br}(\mathbb{D}_{n+1})$  generated by  $\{R_i, E_i\}_{i=3}^{n+1}$  is isomorphic to  $\text{Br}(\mathbb{A}_{n-1})$ , which can be found in [7], or  $\text{Br}(\mathbb{B}_n)/(e_0, r_0 - 1) \cong \text{Br}(\mathbb{A}_{n-1})$ , the proposition below holds naturally.

**Proposition 2.7.** *The subalgebra generated by  $\{r_i, e_i\}_{i=1}^{n-1}$  and  $\delta$  in  $\text{Br}(\mathbb{B}_n)$  is isomorphic to  $\text{Br}(\mathbb{A}_{n-1})$ .*

Hence the formulas in Remark 2.4 still hold for lower letters with nonzero indices. The next two lemmas contain formulas that will be applied in Section 4.

For  $2 \leq i \leq n-1$ , set  $e_1^* = r_0 e_1 r_0$  and  $e_i^* = r_{i-1} r_i e_{i-1}^* r_i r_{i-1}$ .

**Lemma 2.8.** For  $i \in \{1, \dots, n-1\}$ ,

$$e_i e_i^* = e_i e_{i-1} \cdots e_1 e_0 e_1 \cdots e_{i-1} e_i, \quad (2.40)$$

$$r_0 e_i e_i^* = e_i e_i^*. \quad (2.41)$$

*Proof.* For  $i = 1$ , we have  $e_1 e_1^* = e_1 r_0 e_1 r_0 \stackrel{(2.23)}{=} e_1 e_0 e_1$ . For  $i > 1$  induction, (2.10), and (2.1) give  $r_{i-1} r_i e_{i-1} r_i r_{i-1} = e_i$ , so

$$\begin{aligned} e_i e_i^* &= r_{i-1} r_i e_{i-1} r_i r_{i-1} r_{i-1} r_i e_{i-1}^* r_i r_{i-1} \stackrel{(2.1)}{=} r_{i-1} r_i e_{i-1} e_{i-1}^* r_i r_{i-1} \\ &= (r_{i-1} r_i e_{i-1}) \cdots e_1 e_0 e_1 \cdots (e_{i-1} r_i r_{i-1}) \\ &\stackrel{(2.9)+(2.33)}{=} e_i e_{i-1} \cdots e_1 e_0 e_1 \cdots e_{i-1} e_i. \end{aligned}$$

This establishes (2.40). Equality (2.41) follows from (2.40), (2.6), and (2.20).  $\square$

Put

$$g = e_2 r_1 e_0 e_1 e_2. \quad (2.42)$$

**Lemma 2.9.** For  $n \geq 3$ , the following equations hold in  $\text{Br}(\mathbf{B}_n)$ ,

$$g = g^{\text{op}}, \quad (2.43)$$

$$(r_1 r_0 r_1) e_2 r_1 e_0 e_1 = e_2 r_1 e_0 e_1, \quad (2.44)$$

$$(r_1 r_0 r_1) g = g, \quad (2.45)$$

$$e_0 g = \delta e_0 e_2. \quad (2.46)$$

For  $n \geq 4$ , the following equations hold in  $\text{Br}(\mathbf{B}_n)$ ,

$$(r_3 r_2 r_1 r_0 r_1 r_2 r_3) g = g, \quad (2.47)$$

$$e_0 r_1 r_2 r_3 g = \delta e_0 e_1 e_3 r_2 r_3, \quad (2.48)$$

$$e_0 r_1 g = \delta e_0 r_2 r_1 e_2, \quad (2.49)$$

$$e_1 r_2 r_3 g = e_1 e_0 r_1 r_2 r_3 e_2. \quad (2.50)$$

*Proof.* From

$$g = (e_2 r_1) e_0 e_1 e_2 \stackrel{(2.6)}{=} e_2 e_1 (r_2 e_0) e_1 e_2 \stackrel{(2.6)}{=} e_2 e_1 e_0 (r_2 e_1 e_2) \stackrel{(2.34)}{=} e_2 e_1 e_0 r_1 e_2,$$

it follows that  $g$  is invariant under opposition, and so (2.43). Equality (2.44) follows from

$$\begin{aligned} r_1 r_0 (r_1 e_2 r_1) e_0 e_1 &\stackrel{(2.10)}{=} r_1 (r_0 r_2) e_1 (r_2 e_0) e_1 \stackrel{(2.6)+(2.7)}{=} r_1 r_2 (r_0 e_1 e_0) r_2 e_1 \\ &\stackrel{(2.20)}{=} r_1 r_2 e_1 (e_0 r_2) e_1 \stackrel{(2.7)}{=} (r_1 r_2 e_1 r_2) e_0 e_1 \stackrel{(2.10)}{=} e_2 r_1 e_0 e_1. \end{aligned}$$

Equality (2.45) follows from (2.44) by right multiplication by  $e_2$ , and Equality (2.47) from

$$\begin{aligned} r_3 r_2 r_1 r_0 r_1 r_2 r_3 g &\stackrel{(2.9)}{=} r_3 r_2 (r_1 r_0 r_1 e_3) g \stackrel{(2.6)}{=} r_3 r_2 e_3 r_1 r_0 r_1 g \\ &\stackrel{(2.45)}{=} r_3 r_2 e_3 g \stackrel{(2.9)+(2.37)}{=} g. \end{aligned}$$

Formula (2.46) follows from

$$e_0 g = (e_0 e_2) r_1 e_0 e_1 e_2 \stackrel{(2.7)}{=} e_2 (e_0 r_1 e_0) e_1 e_2 \stackrel{(2.15)}{=} \delta e_2 e_0 e_1 e_2 \stackrel{(2.7)+(2.37)}{=} \delta e_0 e_2,$$

Formula (2.48) from

$$\begin{aligned} e_0 r_1 r_2 r_3 g &= e_0 r_1 (r_2 r_3 e_2) r_1 e_0 e_1 e_2 \stackrel{(2.9)}{=} e_0 (r_1 e_3) e_2 r_1 e_0 e_1 e_2 \\ &\stackrel{(2.6)}{=} e_0 e_3 (r_1 e_2 r_1) e_0 e_1 e_2 \stackrel{(2.10)}{=} (e_0 e_3 r_2) e_1 (r_2 e_0) e_1 e_2 \\ &\stackrel{(2.6)+(2.7)}{=} e_3 r_2 (e_0 e_1 e_0) r_2 e_1 e_2 \stackrel{(2.16)}{=} \delta (e_3 r_2 e_0 r_2 e_1) e_2 \\ &\stackrel{(2.1)+(2.6)+(2.7)}{=} \delta e_0 e_1 e_3 e_2 \stackrel{(2.33)}{=} \delta e_0 e_1 e_3 r_2 r_3. \end{aligned}$$

Formula (2.49) from

$$\begin{aligned} e_0 r_1 g &= e_0 (r_1 e_2 r_1) e_0 e_1 e_2 \stackrel{(2.10)}{=} (e_0 r_2) e_1 (r_2 e_0) e_1 e_2 \stackrel{(2.6)}{=} r_2 (e_0 e_1 e_0) r_2 e_1 e_2 \\ &\stackrel{(2.16)}{=} \delta (r_2 e_0 r_2) e_1 e_2 \stackrel{(2.1)+(2.6)}{=} \delta e_0 (e_1 e_2) \stackrel{(2.9)}{=} \delta e_0 r_2 r_1 e_2, \end{aligned}$$

and Formula (2.50) from

$$\begin{aligned} e_1 r_2 r_3 g &= e_1 (r_2 r_3 e_2) r_1 e_0 e_1 e_2 \stackrel{(2.9)}{=} (e_1 e_3) e_2 r_1 e_0 e_1 e_2 \stackrel{(2.7)}{=} e_3 (e_1 e_2 r_1) e_0 e_1 e_2 \\ &\stackrel{(2.36)}{=} (e_3 e_1 r_2 e_0) e_1 e_2 \stackrel{(2.6)+(2.7)}{=} e_1 e_0 e_3 (r_2 e_1 e_2) \stackrel{(2.34)}{=} e_1 e_0 (e_3 r_1) e_2 \\ &\stackrel{(2.6)}{=} e_1 e_0 r_1 (e_3 e_2) \stackrel{(2.9)}{=} e_1 e_0 r_1 r_2 r_3 e_2. \end{aligned}$$

□

In order to give the diagram interpretation of monomials of  $\text{Br}(\mathbf{B}_n)$ , we recall the *Brauer diagram algebra of type  $\widehat{\mathbf{D}}_{n+1}$*  from [7]. Divide  $2n+2$  points into two sets  $\{1, 2, \dots, n+1\}$  and  $\{\hat{1}, \hat{2}, \dots, \hat{n}+1\}$  of points in the (real) plane with each set on a horizontal line and point  $i$  above  $\hat{i}$ . An  $n+1$ -connector is a partition on  $2n+2$  points into  $n+1$  disjoint pairs. It is indicated in the plane by a (piecewise linear) curve, called *strand* from one point of the pair to the other. A *decorated  $n+1$ -connector* is an  $n+1$ -connector in which an even number of pairs are labeled 1, and all other pairs are labeled by 0. A pair labeled 1 will be called *decorated*. The decoration of a pair is



represented by a black dot on the corresponding strand. Denote  $T_{n+1}$  the set of all decorated  $n+1$ -connectors. Denote  $T_{n+1}^0$  the subset of  $T_{n+1}$  of decorated  $n+1$ -connectors without decorations and denote  $T_{n+1}^=$  the subset of  $T_{n+1}$  of decorated  $n+1$ -connectors with at least one horizontal strand.

Let  $H$  be the commutative monoid with presentation

$$H = \langle \delta^{\pm 1}, \xi, \theta \mid \xi^2 = \delta^2, \xi\theta = \delta\theta, \theta^2 = \delta^2\theta \rangle = \langle \delta^{\pm 1} \rangle \{1, \xi, \theta\}.$$

A *Brauer diagram* of type  $D_{n+1}$  is the scalar multiple of a decorated  $n$ -connector by an element of  $H$  belonging to  $\langle \delta^{\pm 1} \rangle (T_{n+1} \cup \xi T_{n+1}^= \cup \theta(T_{n+1}^0 \cap T_{n+1}^=))$ . The *Brauer diagram algebra of type  $D_{n+1}$* , denoted  $\text{BrD}(D_{n+1})$ , is the  $\mathbb{Z}[\delta^{\pm 1}]$ -linear span of all Brauer diagrams of type  $D_{n+1}$  with multiplication laws defined in [7, Definition 4.4]. The scalar  $\xi\delta^{-1}$  appears in various products of  $n+1$ -connectors described in [7, Definition 4.4] and two consecutive black dots on a strand are removed. The multiplication is an intricate variation of the multiplication in classical Brauer diagrams, where the points of the bottom of one connector are joined to the points of the top of the other connector, so as to obtain a new connector. In this process, closed strands appear which are turned into scalars by translating them into elements of  $H$  as indicated in Figure 1.

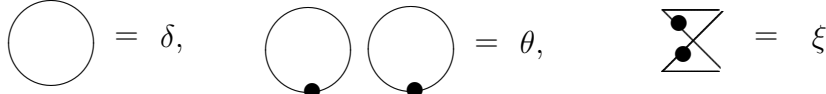


Figure 1: The closed loops corresponding to the generators of  $H$

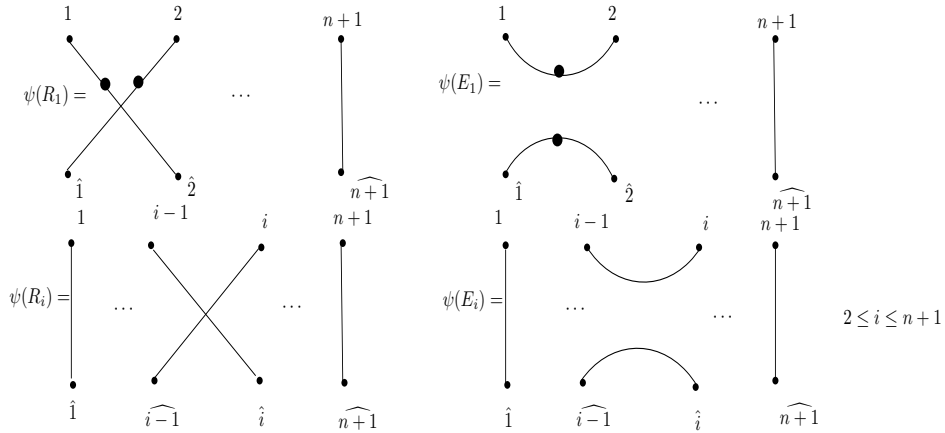


Figure 2: The images of the generators of  $\text{Br}(D_{n+1})$  under  $\psi$

In [7], the algebra  $\text{BrD}(D_{n+1})$  is proved to be isomorphic to  $\text{Br}(D_{n+1})$  by means of the isomorphism  $\psi : \text{Br}(D_{n+1}) \mapsto \text{BrD}(D_{n+1})$  defined on generators as in Figure 2. It is free over  $\mathbb{Z}[\delta^{\pm 1}]$  with basis  $T_{n+1} \cup \xi T_{n+1}^= \cup \theta(T_{n+1}^0 \cap T_{n+1}^=)$ .

**Notation 2.10.** Write  $T_{n+1}^|$  for the subset of  $T_{n+1}$  consisting of all  $n+1$ -connectors with a fixed strand from 1 to  $\hat{1}$ , and set  $T_{n+1}^{|=} = T_{n+1}^| \cap T_{n+1}^=$ . It is readily checked that the union of  $\delta^{\mathbb{Z}} T_{n+1}^|$ ,  $\delta^{\mathbb{Z}} \xi T_{n+1}^{|=}$ , and  $\delta^{\mathbb{Z}} \theta(T_{n+1}^= \cap T_{n+1}^0)$  is a submonoid of  $\text{BrD}(D_{n+1})$ ; we denote it by  $\text{BrMD}(B_n)$  and the corresponding algebra over  $\mathbb{Z}[\delta^{\pm 1}]$  by  $\text{BrD}(B_n)$ .

The images of the generators of  $\text{Br}(B_n)$  under  $\psi\phi$  in  $\text{BrD}(D_{n+1})$  lie in  $\text{BrD}(B_n)$ ; they are indicated in Figure 3.

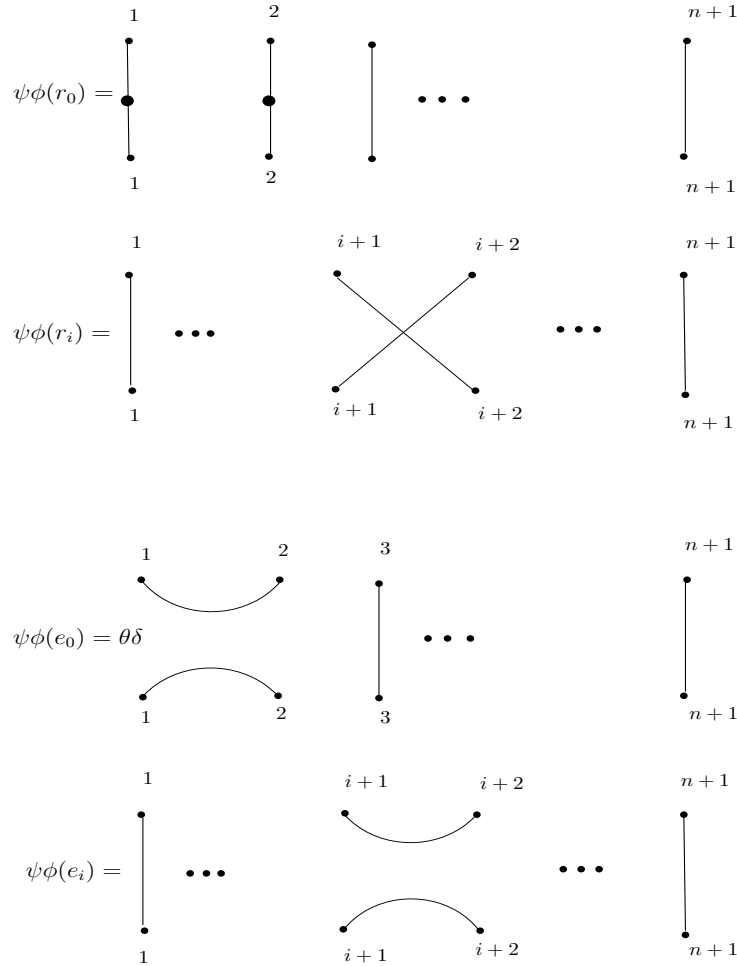


Figure 3: The images under  $\psi\phi$  of the generators of  $\text{Br}(B_n)$

**Theorem 2.11.** *For  $\phi$  and  $\psi$ , the following holds.*

- (i) *The restriction of  $\psi$  to  $\phi(\text{Br}(\text{B}_n))$  is an isomorphism onto  $\text{BrD}(\text{B}_n)$ .*
- (ii) *The image of  $\phi$  coincides with  $\text{SBr}(\text{D}_{n+1})$ .*
- (iii) *The  $\mathbb{Z}[\delta^{\pm 1}]$ -algebras  $\text{SBr}(\text{D}_{n+1})$  and  $\text{BrD}(\text{B}_n)$  are free of rank  $f(n)$ .*

These assertions imply the commutativity of the following diagram.

$$\begin{array}{ccccc}
 \text{Br}(\text{B}_n) & \longrightarrow & \text{SBr}(\text{D}_{n+1}) & \xrightarrow{\cong} & \text{BrD}(\text{B}_n) \\
 & \searrow \phi & \downarrow & & \downarrow \\
 & & \text{Br}(\text{D}_{n+1}) & \xrightarrow[\cong]{} & \text{BrD}(\text{D}_{n+1})
 \end{array}$$

*Proof.* (i). All of the images of the generators  $\{r_i, e_i\}_{i=0}^{n-1}$  under  $\psi\phi$  are in  $\text{BrD}(\text{B}_n)$ . Therefore, the assertion is equivalent to the statement that all elements in  $\text{BrMD}(\text{B}_n)$  can be written as products of  $\{\psi\phi(r_i), \psi\phi(e_i)\}_{i=0}^{n-1}$  up to some powers of  $\delta$ . Before we start to verify this fact, we observe that the two special elements  $K_i$  and  $E_{i,j}$  in  $\text{BrD}(\text{B}_n)$  of Figure 2 are both in  $\psi\phi(\text{Br}(\text{B}_n))$ .

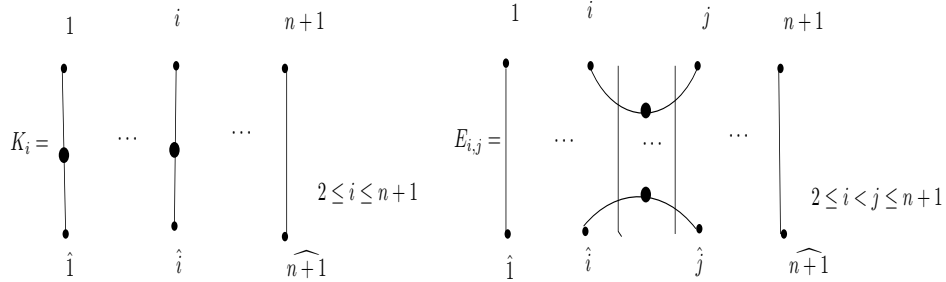


Figure 4:  $K_i$  and  $E_{i,j}$

This can be verified by induction on  $i$  and  $j$  (alternatively, they are  $\psi\phi(r_{\epsilon_i})$  and  $\psi\phi(e_{\alpha_j + \alpha_i})$  in the notation of section 3).

Let  $a \in \text{BrMD}(\text{B}_n)$ . We will show  $a \in \psi\phi(\text{Br}(\text{B}_n))$ . For  $n = 2, 3$ , the requires assertion can be checked by use of diagrams. We proceed by induction on  $n$  and let  $n > 3$ .

First assume  $a \in \delta^{\mathbb{Z}} T_{n+1}^{|} \cup \delta^{\mathbb{Z}} \xi T_{n+1}^{|=}$ . Notice that  $a$  has a vertical strand from 1 to  $\hat{1}$ . If  $a$  has another vertical strand, say between  $i$  and  $\hat{j}$  with

$1 < i, j \leq n+1$ , then, by multiplying  $a$  by a suitable element of  $\langle \psi\phi(r_i) \rangle_{i=1}^{n-1}$  (which is isomorphic to  $W(A_{n-1})$ ) at the left and by another element of it at the right, we can move this vertical strand so that it connects  $n+1$  and  $\widehat{n+1}$ . If it is decorated then we apply  $K_{n+1}$  (an element in the image of  $\psi\phi$ ) to remove the decoration on this strand; as a consequence, we may restrict ourselves to  $a \in \text{BrD}(B_{n-1})$ . But then induction applies and gives  $a \in \psi\phi(\text{Br}(B_n))$ .

If  $a$  has only one vertical strand, then the strands from  $n+1$  and  $\widehat{n+1}$  are horizontal. As above, we multiply by two suitable elements of  $\langle \psi\phi(r_i) \rangle_{i=1}^{n-1}$ , which is isomorphic to  $W(A_{n-1})$ , to move the top horizontal strand to the strand connecting  $\widehat{n}$  and  $n+1$  and the bottom horizontal strand to one connecting  $\widehat{n}$  and  $\widehat{n+1}$ . Next we apply  $K_{n+1}$  to remove possible decorations on the two strands. As a result,  $a \in \text{Br}(B_{n-2})$  and so  $a \in \psi\phi(\text{Br}(B_n))$  by the induction hypothesis.

The case  $a \in \delta^{\mathbb{Z}}\theta(T_{n+1}^= \cap T_{n+1}^0)$  remains. We distinguish five possible cases for  $a$  by the strands with ends 1 and  $\hat{1}$ .

- $M^{(2)}$  is the subset of  $T_{n+1}^= \cap T_{n+1}^0$  of all diagrams with a fixed vertical strand between 1 and  $\hat{1}$ ,
- $M^{(3)}$  is the subset of  $T_{n+1}^= \cap T_{n+1}^0$  of all diagrams with two different horizontal strands with ends 1 and  $\hat{1}$ ,
- $M^{(4)}$  is the subset of  $T_{n+1}^= \cap T_{n+1}^0$  of all diagrams with two different vertical strands with ends 1 and  $\hat{1}$ ,
- $M^{(5)}$  is the subset of  $T_{n+1}^= \cap T_{n+1}^0$  of all diagrams with a horizontal strand starting from 1 and a vertical strand from  $\hat{1}$ , and
- $M^{(6)}$  is the subset of  $T_{n+1}^= \cap T_{n+1}^0$  of all diagrams with a horizontal strand starting from  $\hat{1}$  and vertical strands from 1.

If  $a \in \delta^{\mathbb{Z}}\theta M^{(2)}$ , the diagram part can be written as a scalar multiple of the image of some element  $b \in \delta^{\mathbb{Z}}\psi\phi\langle r_i, e_i \rangle_{i=1}^{n-1}$ , so  $a = \delta^k\theta\psi\phi(b)$  for some  $k \in \mathbb{Z}$ . If there is a horizontal strand at the top between  $i$  and  $j$ , where  $1 < i < j \leq n+1$ , then  $a = \delta^k E_{i,j}\psi\phi(b)$  for some  $k \in \mathbb{Z}$ , and we are done as  $E_{i,j}$  lies in the image of  $\psi\phi$ .

If  $a \in \delta^{\mathbb{Z}}M^{(5)}$  (or  $M^{(6)}$ ,  $M^{(4)}$ ,  $M^{(3)}$ , respectively), then, by multiplying by suitable elements in  $\psi\phi\langle r_i \rangle_{i=1}^{n-1}$  (which is isomorphic to  $W(A_{n-1})$ ) at both sides of  $a$ , we can achieve that the strands between  $\{i, \hat{i}\}_{i=1}^3$  are as in the diagram of  $\psi\phi(e_0e_1)$  (or  $\{i, \hat{i}\}_{i=1}^3$  as in  $\psi\phi(e_1e_0)$ ,  $\{i, \hat{i}\}_{i=1}^4$  as in  $\psi\phi(e_2r_1e_0e_1e_2)$ ,  $\{i, \hat{i}\}_{i=1}^2$  as in  $\psi\phi(e_0)$ , respectively). Up to the leftmost 3 or 4 strands, the resulting diagram can be considered as an element of  $\delta^{\mathbb{Z}}\psi\phi\langle r_i, e_i \rangle_{i=3}^{n-1}$  (or

$\delta^{\mathbb{Z}}\psi\phi\langle r_i, e_i\rangle_{i=3}^{n-1}$ ,  $\delta^{\mathbb{Z}}\psi\phi\langle r_i, e_i\rangle_{i=4}^{n-1}$ ,  $\delta^{\mathbb{Z}}\psi\phi\langle r_i, e_i\rangle_{i=2}^{n-1}$ , respectively) which is isomorphic to  $\text{BrM}(A_j)$  for  $j = n - 3$  (respectively,  $j = n - 3, n - 4, n - 2$ ). Therefore, the first claim (i) holds.

(ii). It follows from (i) that

$$\psi^{-1}(\text{BrD}(B_n)) = \phi(\text{Br}(B_n)) \subseteq \text{SBr}(D_{n+1}).$$

Therefore, it suffices to prove  $\text{SBrM}(D_{n+1}) \subseteq \phi(\text{BrM}(B_n))$ , or, equivalently,

$$(\text{BrMD}(D_{n+1}) \setminus \text{BrMD}(B_n)) \cap \psi(\text{SBrM}(D_{n+1})) = \emptyset.$$

We find that  $\text{BrMD}(D_{n+1}) \setminus \text{BrMD}(B_n)$  consists of  $\delta^{\mathbb{Z}}M^{(i)} \cup \xi\delta^{\mathbb{Z}}M^{(i)}$ , for  $i = 3, \dots, 6$ . By an argument analogous to the above and subsequently multiplying by  $K_i$  to remove decorations on all strands except the leftmost 3 or 4 strands, we can reduce the verification to a case where  $n = 2, 3$ , and so finish by induction.

(iii). By (i) and (ii), the algebras  $\text{BrD}(B_n)$  and  $\text{SBr}(D_{n+1})$  are isomorphic  $\mathbb{Z}[\delta^{\pm 1}]$ -algebras. The latter is free (as stated above) and the definition of the former shows that its rank is  $|T_{n+1}^{\downarrow}| + |T_{n+1}^{\downarrow=}| + |T_n^{\downarrow} \cap T_{n+1}^0|$ . A simple counting argument gives  $|T_{n+1}^{\downarrow}| = 2^n \cdot n!!$ ,  $|T_{n+1}^{\downarrow=}| = 2^n \cdot n!! - n!$ , and  $|T_n^{\downarrow} \cap T_{n+1}^0| = (n+1)!! - (n+1)!$ . We conclude that the rank of  $\text{BrD}(B_n)$  is equal to  $f(n)$ .  $\square$

For  $n = 3$ , the eight Brauer diagrams in  $\psi\phi(F)$  of  $\text{BrD}(D_4)$  ([7, Section 4]) are depicted in Figure 5.

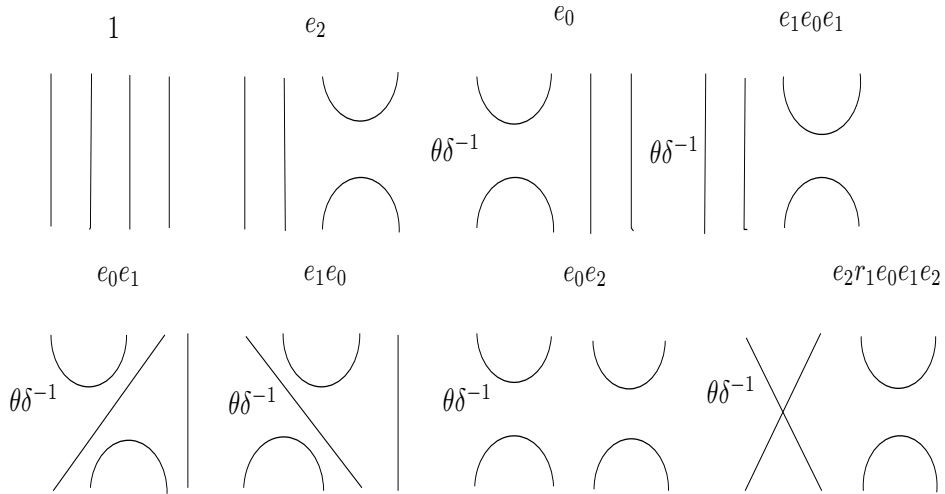


Figure 5: The images of  $F$  under  $\psi\phi$

### 3 Admissible root sets and the monoid action

In this section, we recall some facts about the root system associated to the Brauer algebra of type  $D_{n+1}$ , introduce the definition of admissible root sets of type  $B_n$  and describe some of its basic properties. Also by use of [8], we give a monoid action of  $\text{BrM}(B_n)$  on the admissible root sets.

**Definition 3.1.** By  $\Phi$  and  $\Phi^+$ , we denote the root system of type  $D_{n+1}$  and its positive roots, and  $\Phi^+$  can be realized by vectors  $\epsilon_j \pm \epsilon_i$  with  $j > i$  in  $\mathbb{R}^{n+1}$  where  $\{\epsilon_i\}_{i=1}^{n+1}$  are the canonical orthonormal basis. The simple roots are  $\alpha_1 = \epsilon_1 + \epsilon_2$  and  $\alpha_i = \epsilon_i - \epsilon_{i-1}$  for  $i = 2, \dots, n+1$ . If  $\alpha$  is a root of  $\Phi^+$ , then  $\alpha^*$  denotes its *orthogonal mate*, that is,  $\alpha^* = \epsilon_j + \epsilon_i$  if  $\alpha = \epsilon_j - \epsilon_i \in \Phi^+$ , and  $(\alpha^*)^* = \alpha$ . When  $n \geq 4$ , this is the unique positive root orthogonal to  $\alpha$  and all other positive roots orthogonal to  $\alpha$  (see [6, Definition 2.6]). The diagram automorphism  $\sigma$  on  $W(D_{n+1})$  is induced by a linear isomorphism  $\sigma$  on  $\mathbb{R}^{n+1}$ , where  $\sigma$  is the orthogonal reflection with root  $\epsilon_1$ . We define  $\mathfrak{p} : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  by  $\mathfrak{p}(x) = (\sigma(x) + x)/2$ . The set  $\Psi = \mathfrak{p}(\Phi)$  forms a root system of the Weyl group of type  $B_n$  with  $\beta_0 = \epsilon_2 = (\alpha_1 + \alpha_2)/2$  and  $\beta_i = \epsilon_{i+2} - \epsilon_{i+1} = \alpha_{i+2}$  for  $i = 1, \dots, n-1$  as simple roots, and  $\Psi^+ = \mathfrak{p}(\Phi^+)$  consists of all positive roots of the Weyl group of  $W(B_n)$ . A root  $\beta \in \Psi$  is called a *long root* if  $|\beta| = \sqrt{2}$ , and called a *short root* if  $|\beta| = 1$ . The subset of  $\Psi^+$  consisting of all short roots is  $\{\epsilon_i\}_{i=2}^{n+1}$ .

A set of mutually orthogonal positive roots  $A \subset \Phi^+$  is called *admissible* if, whenever  $\gamma_1, \gamma_2, \gamma_3$  are distinct roots in  $A$  and there exists a root  $\gamma \in \Phi$  for which  $|(\gamma, \gamma_i)| = 1$  for all  $i$ , the positive root of  $\pm R_\gamma R_{\gamma_1} R_{\gamma_2} R_{\gamma_3} \gamma$  is also in  $A$ . An equivalent definition states that either there is no orthogonal mate for any  $\alpha \in A$  or for each  $\alpha \in A$  we have  $\alpha^* \in A$ .

A subset  $B$  of  $\Psi^+$  of mutually orthogonal roots is called *admissible*, if  $\mathfrak{p}^{-1}(B) \cap \Phi \subset \Phi^+$  is admissible. This is equivalent to  $B$  being the image of some  $\sigma$ -invariant admissible set of  $\Phi^+$ .

We write  $\mathcal{A}$  for the collection  $\mathcal{A}$  of admissible root subsets of  $\Phi^+$  and  $\mathcal{A}_\sigma$  for the set of all admissible subsets of  $\Psi^+$  left invariant by  $\sigma$ . Its elements correspond to the  $\sigma$ -invariant elements in  $\mathcal{A}$  via  $B \mapsto \mathfrak{p}^{-1}(B)$ .

*Remark 3.2.* The mutually orthogonal root set  $B_1 = \{\epsilon_2, \epsilon_3\}$  is not admissible, for  $\mathfrak{p}^{-1}(B_1) \cap \Phi = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$  is not admissible. Although  $B_2 = \{\alpha_1, \alpha_4\}$  is admissible in  $\Phi^+$ , the set  $\mathfrak{p}(B_2)$  is not admissible, as  $\mathfrak{p}^{-1}\mathfrak{p}(B_2) = \{\alpha_1, \alpha_2, \alpha_4\}$  is not admissible.

By use of the description of admissible sets of type D in [6, Section 4], the admissible sets of type B can be classified as indicated below.

**Proposition 3.3.** *The admissible root subsets of  $\Psi^+$  can be divided into the following three types.*

- (1) The set consists of long roots and none of their orthogonal mates are in this set. It belongs to the  $W(B_n)$ -orbit of  $Z_t = \{\beta_{n-1-2i} \mid 0 \leq i < t\}$ , for some  $t$  with  $0 \leq t < (n+1)/2$ .
- (2) The set consists of long roots and their orthogonal mates. It belongs to the  $W(B_n)$ -orbit of  $\tilde{Z}_t = \{\beta_{n-1-2i}, \beta_{n-1-2i}^* \mid 0 \leq i < t\}$ , for some  $t$  with  $1 \leq t < (n+1)/2$ .
- (3) The set has only one short root, and for each long root that it also contains, it also contains its orthogonal mate. It belongs to the  $W(B_n)$ -orbit of  $\bar{Z}_t = \{\beta_{n-1-2i}, \beta_{n-1-2i}^* \mid 0 \leq i < t-1\} \cup \{\beta_0\}$ , for some  $t$  with  $1 \leq t \leq (n+1)/2$ .

**Lemma 3.4.** *The cardinalities of the  $W(B_n)$ -orbits of  $Z_t$ ,  $\bar{Z}_t$ , and  $\tilde{Z}_t$  in the above are  $2^t \binom{n}{2t} t!!$ ,  $n \binom{n-1}{2t-2} (t-1)!!$ , and  $\binom{n}{2t} t!!$ , respectively.*

*Proof.* The orbits of  $Z_t$ ,  $\bar{Z}_t$ , and  $\tilde{Z}_t$  correspond to, respectively,

- (1) the diagrams with exactly  $t$  decorated horizontal strands at the top in  $\text{BrD}(D_{n+1})$  none of which has end 1,
- (2) the diagrams with exactly  $t$  horizontal strands at the top without decoration one of which has end point 1,
- (3) the diagrams with exactly  $t$  horizontal strands at the top without decoration and none of which has end point 1.

Therefore, the sizes are easily seen to be as stated.  $\square$

Just as in [4], these results will be applied to compute the rank of  $\text{Br}(B_n)$ . The lemma below can be proved analogously to Lemma 5.7 of [8].

**Lemma 3.5.** *Let  $i$  and  $j$  be nodes of the Dynkin diagram  $B_n$ . If  $w \in W(B_n)$  satisfies  $w\beta_i = \beta_j$ , then  $we_iw^{-1} = e_j$ .*

Consider a positive root  $\beta$  and a node  $i$  of type  $B_n$ . If there exists  $w \in W$  such that  $w\beta_i = \beta$ , then we can define the element  $e_\beta$  in  $\text{BrM}(B_n)$  by  $e_\beta = we_iw^{-1}$ . The above lemma implies that  $e_\beta$  is well defined. In general,  $we_\beta w^{-1} = e_{w\beta}$ , for  $w \in W(C_{n+1})$  and  $\beta$  a root of  $W(B_n)$ . Note that  $e_\beta = e_{-\beta}$  in view of (2.2). We write  $e_\beta^* = e_{\beta^*}$  and  $e_i^* = e_{\beta_i^*}$  for  $\beta \in \Psi^+$  and  $1 \leq i \leq n-1$  in accordance with the definition before Lemma 2.8.

**Lemma 3.6.** *If  $\{\gamma_1, \gamma_2\}$  is a subset of some admissible root set, then*

$$e_{\gamma_1} e_{\gamma_2} = e_{\gamma_2} e_{\gamma_1}.$$

*Proof.* In view of Proposition 3.3, the proof can be reduced to a check in the following three cases

$$(\gamma_1, \gamma_2) = (\beta_0, \beta_2), (\beta_1, \beta_3), \text{ or } (\beta_1, \beta_1^*).$$

In the first two cases the lemma holds due to (2.7). In the third case it holds by (2.13) and (2.23).  $\square$

Let  $X \subset \Psi^+$  be a subset of some admissible root set. Then we define

$$e_X = \prod_{\beta \in X} e_\beta. \quad (3.1)$$

In view of Lemma 3.6, it is well defined. Since the intersection of admissible sets in  $\mathcal{A}$  is still an admissible set, there is an admissible closure for mutually orthogonal subset of  $\Phi^+$ , we can similarly give a definition for some subsets of  $\Psi^+$  as the following.

**Definition 3.7.** Suppose that  $X \subset \Psi^+$  is a mutually orthogonal root set. If  $X$  is a subset of some admissible root set, then the minimal admissible set containing  $X$  is called the *admissible closure* of  $X$ , denoted by  $\overline{X}$ .

**Lemma 3.8.** Let  $X \subset \Psi^+$  be a mutually orthogonal root set and  $\overline{X}$  exists. Then

$$e_{\overline{X}} = \delta^{|\overline{X} \setminus X|} e_X.$$

*Proof.* If  $\overline{X}$  is in the  $W(B_n)$ -orbit of  $Z_t$ , then it is trivial. If  $\overline{X}$  is in the  $W(B_n)$ -orbit of  $\tilde{Z}_t$  or  $\tilde{\tilde{Z}}_t$ , then  $X$  can be transformed into a subset of  $\tilde{Z}_t$  or  $\tilde{\tilde{Z}}_t$  by the action of some element of  $W(B_n)$ . Hence we just consider subsets of  $\tilde{Z}_t$  and  $\tilde{\tilde{Z}}_t$ . Now

$$\begin{aligned} e_0 e_2 e_2^* &= e_0 e_2 r_1 r_0 (r_1 e_2 r_1) r_0 r_1 \stackrel{(2.10)}{=} e_0 e_2 r_1 (r_0 r_2) e_1 r_2 r_0 r_1 \\ &\stackrel{(2.5)}{=} e_0 (e_2 r_1 r_2) r_0 e_1 (r_2 r_0) r_1 \stackrel{(2.33)+(2.5)}{=} (e_0 e_2) (e_1 r_0 e_1 r_0) r_2 r_1 \\ &\stackrel{(2.7)+(2.23)}{=} e_2 (e_0 e_1 e_0) e_1 r_2 r_1 \stackrel{(2.16)}{=} \delta (e_2 e_0) e_1 r_2 r_1 \\ &\stackrel{(2.7)}{=} \delta e_0 (e_2 e_1 r_2 r_1) \stackrel{(2.36)+(2.1)}{=} \delta e_0 e_2, \end{aligned}$$



$$\begin{aligned}
e_1 e_1^* e_3 e_3^* &\stackrel{(2.23)}{=} e_1 e_0 (e_1 e_3) r_2 r_1 r_0 r_1 r_2 e_3 r_2 r_1 r_0 r_1 r_2 \\
&\stackrel{(2.7)}{=} e_1 e_0 e_3 (e_1 r_2 r_1) r_0 r_1 r_2 e_3 r_2 r_1 r_0 r_1 r_2 \\
&\stackrel{(2.33)}{=} e_1 e_0 e_3 e_1 (e_2 r_0) r_1 r_2 e_3 r_2 r_1 r_0 r_1 r_2 \\
&\stackrel{(2.6)}{=} e_1 e_0 (e_3 e_1 r_0) (e_2 r_1 r_2) e_3 r_2 r_1 r_0 r_1 r_2 \\
&\stackrel{(2.6)+(2.7)+(2.33)}{=} e_1 e_0 e_1 r_0 e_3 e_2 (e_1 e_3) r_2 r_1 r_0 r_1 r_2 \\
&\stackrel{(2.18)}{=} e_1 e_0 e_1 e_3 e_2 e_3 (e_1 r_2 r_1) r_0 r_1 r_2 \\
&\stackrel{(2.33)}{=} e_1 e_0 e_1 (e_3 e_2 e_3) e_1 e_2 r_0 r_1 r_2 \\
&\stackrel{(2.37)}{=} e_1 e_0 e_1 (e_3 e_1) e_2 r_0 r_1 r_2 \stackrel{(2.7)}{=} e_1 e_0 (e_1 e_1) e_3 e_2 r_0 r_1 r_2 \\
&\stackrel{(2.3)}{=} \delta e_1 e_0 e_1 (e_3 e_2 r_0) r_1 r_2 \stackrel{(2.6)}{=} \delta (e_1 e_0 e_1 r_0) e_3 e_2 r_1 r_2 \\
&\stackrel{(2.18)}{=} \delta e_1 e_0 (e_1 e_3) e_2 r_1 r_2 \stackrel{(2.7)}{=} \delta e_1 e_0 e_3 (e_1 e_2 r_1 r_2) \\
&\stackrel{(2.36)}{=} \delta e_1 e_0 e_3 e_1 (r_2 r_2) \stackrel{(2.1)}{=} \delta e_1 e_0 e_3 e_1 \stackrel{(2.23)}{=} \delta e_1 e_1^* e_3.
\end{aligned}$$

This proves the lemma for  $|X| = 2, 3$ . By applying induction on  $|X|$ , we see the lemma holds.  $\square$

*Remark 3.9.* In [4, Definition 3.2], an action of the Brauer monoid  $\text{BrM}(\text{D}_{n+1})$  on the collection  $\mathcal{A}$  is defined as follows. The generators  $\{R_i\}_{i=1}^{n+1}$  act by the natural action of Coxeter group elements on its root sets, where negative roots are negated so as to obtain positive roots, the element  $\delta$  acts as the identity, and the action of  $\{E_i\}_{i=1}^{n+1}$  is defined below.

$$E_i B := \begin{cases} B & \text{if } \alpha_i \in B, \\ \overline{B \cup \{\alpha_i\}} & \text{if } \alpha_i \perp B, \\ R_\beta R_i B & \text{if } \beta \in B \setminus \alpha_i^\perp. \end{cases} \quad (3.2)$$

In [7, Section 4], a root  $\epsilon_i - \epsilon_j$  ( $\epsilon_i + \epsilon_j$ ) with  $1 \leq j < i \leq n+1$  is represented as a (decorated) horizontal strand from  $i$  to  $j$  at the top of the diagram. This way, the above monoid action can be given a diagram explanation in which admissible sets are represented by the set of horizontal strands at the top of a diagram.

As in [8, Proposition 5.6], there is a monoid action of  $\text{BrM}(\text{B}_n)$  on  $\mathcal{A}_\sigma$  under the composition of the above action of  $\text{Br}(\text{D}_{n+1})$  and  $\phi$ . It gives an action of  $\text{Br}(\text{B}_n)$  on the collection of admissible subsets of  $\Psi^+$ . This action also has a diagrammatic interpretation obtained from viewing the admissible subsets of  $\Psi^+$  as tops of symmetric diagrams by use of  $\mathbf{p}^{-1}$ .

## 4 Upper bound on the rank

In Theorem 4.8 of this section, normal forms for elements of the Brauer monoid  $\text{BrM}(\mathbf{B}_n)$  will be given, and in Corollary 4.18, a spanning set for  $\text{Br}(\mathbf{B}_n)$  of size  $f(n)$  will be given. The rank of  $\text{Br}(\mathbf{B}_n)$  will be proved to be  $f(n)$  as a consequence. These results will provide a proof of Theorem 1.2.

The normal forms of monomials in  $\text{Br}(\mathbf{B}_n)$  will be parameterized by a set  $F$  of elements  $f_t^{(i)}$  to be defined below, a general form being  $a = uf_t^{(i)}v$  for certain elements  $u, v \in W(\mathbf{B}_n)$ . In fact,  $F$  is a set of representatives for the  $W(\mathbf{B}_n)$ -orbits of the admissible sets  $a(\emptyset)$  and  $a^{\text{op}}(\emptyset)$  which appear in the guise of horizontal strands at top and bottom, respectively, as discussed in Remark 3.9. Recall  $g = e_2r_1e_0e_1e_2$  from (2.42).

**Notation 4.1.** We define

$$\begin{aligned} f_t^{(1)} &:= e_{Z_t} = \prod_{i=1}^t e_{n+1-2i}, & 0 \leq t < (n+1)/2, \\ f_t^{(2)} &:= e_{\tilde{Z}_t} = \prod_{i=1}^t e_{n+1-2i} e_{n+1-2i}^*, & 1 \leq t \leq n/2, \\ f_t^{(3)} &:= e_{\tilde{Z}_t}, & 1 \leq t \leq (n+1)/2, \\ f_t^{(4)} &:= gf_{t-1}^{(2)}, & 2 \leq t \leq (n-1)/2, \\ f_t^{(5)} &:= e_0e_1f_{t-1}^{(2)}, & 2 \leq t \leq n/2, \\ f_t^{(6)} &:= e_1e_0f_{t-1}^{(2)}, & 2 \leq t \leq n/2, \end{aligned}$$

and  $f_1^{(4)} := g$ ,  $f_1^{(5)} := e_0e_1$ ,  $f_1^{(6)} := e_1e_0$ . Furthermore, we denote by  $F$  the set of all elements  $f_t^{(i)}$ , and write  $M = \delta^{\mathbb{Z}}W(\mathbf{B}_n)FW(\mathbf{B}_n)$ .

The following statement is immediate from the definition of  $M$ .

**Lemma 4.2.** *The set  $M$  is closed under multiplication by any element from  $W(\mathbf{B}_n)$ .*

Note that  $f_t^{(3)} = e_0f_{t-1}^{(2)}$ , for  $2 \leq t \leq [(n+1)/2]$ . The set  $F$  is closed under the natural anti-involution  $x \mapsto x^{\text{op}}$  of Proposition 2.5.

For  $n = 3$ , we find  $f_0^{(1)} = 1$ ,  $f_1^{(1)} = e_2$ ,  $f_1^{(3)} = e_0$ ,  $f_1^{(6)} = e_1e_0$ ,  $f_1^{(5)} = e_0e_1$ ,  $f_1^{(2)} = e_2e_2^* = r_1r_2(e_1e_0e_1)r_2r_1$ ,  $f_2^{(3)} = \delta e_0e_2$ , and  $f_1^{(4)} = g = e_2r_1e_0e_1e_2$ , which fits (up to conjugation by a Coxeter element) with the eight elements of  $F$  depicted in Figure 5.

Part of Theorem 4.8 states that each member of  $M$  has a unique decomposition, the other part states that  $M$  coincides with  $\text{BrM}(\mathbf{B}_n)$ . The first

part, whose proof takes all of this section up to the statement of the theorem, is devoted to giving a normal form for each element of  $M$ , and involves a narrowing down of the possibilities for the elements of  $W(B_n)$  in a normal form at both sides of  $f_t^{(i)}$ . The second part is carried out after the statements of Theorem 4.8, where it is shown that  $M$  is invariant under multiplication by  $r_i$  and  $e_i$  for each  $i \in \{0, \dots, n-1\}$ . As  $F = F^{\text{op}}$ , it suffices to consider multiplication from the left. As  $e_i u f_t^{(i)} = u e_\beta f_t^{(i)}$ , where  $\beta = u^{-1} \beta_i$ , it suffices to verify that  $e_\gamma f_t^{(i)}$  belongs to  $M$  for each  $\gamma \in \Psi^+$ . This is the content of Lemmas 4.9–4.16.

**Definition 4.3.** Let  $X$  be an admissible subset of  $\Psi^+$ . Recall the action of  $W(B_n)$  on all admissible subsets of  $\Psi^+$  as explained in Remark 3.9. The stabilizer of  $X$  in  $W(B_n)$  is denoted by  $N(X)$ . We select a family  $D_X$  of left coset representatives of  $N(X)$  in  $W(B_n)$ , so  $|D_X|$  is the size of  $W(B_n)$ -orbits of  $X$ . We simplify  $D_{Z_t}$ ,  $D_{\bar{Z}_t}$ ,  $D_{\bar{Z}_t}$ ,  $N_{Z_t}, N_{\bar{Z}_t}$ ,  $N_{\bar{Z}_t}$  to  $D_t^{(1)}$ ,  $D_t^{(2)}$ ,  $D_t^{(3)}$ ,  $N_t^{(1)}$ ,  $N_t^{(2)}$ ,  $N_t^{(3)}$  respectively.

In order to identify the part of  $W(B_n)$  that commutes with  $f_t^{(i)}$ , we introduce the following subgroups of  $W(B_n)$ .

$$\begin{aligned} C_0^{(1)} &= W(B_n), \\ C_t^{(1)} &= \langle r_{n-2}^*, r_0, r_1, \dots, r_{n-2-2t} \rangle, \quad 1 \leq t \leq n/2, \\ C_t^{(2)} &= \langle r_1, r_2, \dots, r_{n-2-2t} \rangle, \quad 1 \leq t \leq n/2, \\ C_t^{(3)} &= \langle r_2, r_3, \dots, r_{n-2t} \rangle, \quad 1 \leq t \leq (n+1)/2, \\ C_t^{(4)} &= \langle r_4, r_5, \dots, r_{n-2t} \rangle, \quad 1 \leq t \leq (n-1)/2, \\ C_t^{(5)} = C_t^{(6)} &= \langle r_3, r_4, \dots, r_{n-2t} \rangle, \quad 1 \leq t \leq n/2. \end{aligned}$$

For  $0 \leq t \leq [n/2]$ , write

$$\begin{aligned} A_t^{(1)} &= \langle r_{n-2i} r_{n+1-2i} r_{n-1-2i} r_{n-2i} \rangle_{i=1}^{t-1} \times \langle r_{n+1-2i} \rangle_{i=1}^t, \\ W_t^{(1)} &= \langle r_0, r_1, \dots, r_{n-1-2t} \rangle \times \langle r_{n+1-2i}^* \rangle_{i=1}^t. \end{aligned}$$

For  $1 \leq t \leq [n/2]$ , write

$$\begin{aligned} A_t^{(2)} &= \langle r_{\epsilon_{n+2-2t}}, A_t^{(1)} \rangle \times \langle r_{n+1-2i}^* \rangle_{i=1}^t, \\ W_t^{(2)} &= \langle r_0, r_1, \dots, r_{n-1-2t} \rangle. \end{aligned}$$

For  $1 \leq t \leq [(n+1)/2]$ , write

$$\begin{aligned} A_t^{(3)} &= \langle r_{\epsilon_{n+4-2t}}, r_{n-2i} r_{n+1-2i} r_{n-1-2i} r_{n-2i} \rangle_{i=1}^{t-2} \times \langle r_{n+1-2i}, r_{n+1-2i}^* \rangle_{i=1}^{t-1} \times \langle r_0 \rangle, \\ W_t^{(3)} &= \langle r_1 r_0 r_1, r_2, r_3, \dots, r_{n+1-2t} \rangle. \end{aligned}$$

For  $1 \leq t \leq (n-1)/2$ , write

$$\begin{aligned} A_t^{(4)} &= \left\langle r_1 r_0 r_1, r_{\epsilon_{n+4-2t}-\epsilon_4} r_2 r_{n+3-2t} r_{\epsilon_{n+4-2t}-\epsilon_4}, A_{t-1}^{(1)}, r_2, r_2^*, \{r_{n+1-2i}, r_{n+1-2i}^*\}_{i=1}^{t-1}, r_0 \right\rangle, \\ W_t^{(4)} &= \left\langle r_{\epsilon_5}, C_t^{(4)} \right\rangle. \end{aligned}$$

For  $1 \leq t \leq n/2$ , write

$$\begin{aligned} A_t^{(5)} &= A_t^{(3)} \times \langle r_1 r_0 r_1 \rangle, \\ W_t^{(5)} &= \langle r_2 r_1 r_0 r_1 r_2, r_3, r_4, \dots, r_{n+1-2t} \rangle. \end{aligned}$$

We first determine the structure of these subgroups.

**Lemma 4.4.** (i) For  $1 \leq t \leq [n/2]$ , we have

$$\begin{aligned} C_t^{(1)} &\cong W(B_{n-2t}) \times W(A_1), \\ A_t^{(1)} &\cong W(A_{t-1}) \times (W(A_1))^t, \\ W_t^{(1)} &\cong W(B_{n-2t})(W(A_1))^t. \end{aligned}$$

(ii) For  $1 \leq t \leq [n/2]$ , we have

$$\begin{aligned} C_t^{(2)} &\cong W(A_{n-1-2t}), \\ A_t^{(2)} &\cong W(B_t) \times (W(A_1))^{2t}, \\ W_t^{(2)} &\cong W(B_{n-2t}). \end{aligned}$$

(iii) For  $1 \leq t \leq [(n+1)/2]$ , we have

$$\begin{aligned} C_t^{(3)} &\cong W(A_{n-2t}), \\ A_t^{(3)} &\cong W(B_{t-1}) \times (W(A_1))^{2t-1}, \\ W_t^{(3)} &\cong W(B_{n+1-2t}). \end{aligned}$$

(iv) For  $1 \leq t \leq (n-1)/2$ , we have

$$\begin{aligned} A_t^{(4)} &\cong W(B_t) \times W(A_1)^{2t+1}, \\ W_t^{(4)} &\cong W(B_{n-1-2t}), \\ A_t^{(4)} \cap W_t^{(4)} &= \{1\}, \\ C_t^{(4)} &\cong W(A_{n-2-2t}). \end{aligned}$$

Furthermore, the group  $A_t^4$  normalizes  $W_t^4$ .

(v) For  $1 \leq t \leq n/2$ , we have

$$\begin{aligned} A_t^{(5)} &\cong W(B_{t-1}) \times (W(A_1))^{2t}, \\ W_t^{(5)} &\cong W(B_{n-2t}), \\ A_t^{(5)} \cap W_t^{(5)} &= \{1\}, \\ C_t^{(5)} &\cong W(A_{n-1-2t}). \end{aligned}$$

Furthermore,  $A_t^{(5)}$  normalizes  $W_t^{(5)}$ .

*Proof.* With the diagram representation of Figure 2, the argument is analogous to [8, Lemma 6.4].  $\square$

**Lemma 4.5.** For  $i = 1, 2, 3$ , the subgroup  $N_t^{(i)}$  in  $W(B_n)$  is a semiproduct of  $W_t^{(i)}$  and  $A_t^{(i)}$ .

*Proof.* The semidirect group structure of these subgroups can be proved by use of the diagram representation of Figure 2. The normalizer (stabilizer) claims follow from Lagrange's Theorem and Lemma 3.4.  $\square$

**Definition 4.6.** We need a few more subgroups and coset representatives. For  $i = 4, 5$ , let  $N_t^{(i)} = A_t^{(i)} W_t^{(i)}$ , and  $D_t^{(i)}$  be its left coset representatives in  $W(B_n)$ . The sets  $\tilde{Z}_t$  and  $\tilde{Z}_{t-1} \cup \{\beta_1, \beta_1^*\}$  are conjugate by  $\tau = r_{\epsilon_{n+2-2t-\epsilon_3}} r_1 r_{n+1-2t} r_{\epsilon_{n+2-2t-\epsilon_3}}$ , and  $\tau C_t^{(2)} \tau^{-1} = C_t^{(6)}$ . At the same time we have  $\tau f_t^{(2)} \tau^{-1} = \delta e_1 f_{t-1}^{(2)}$ . Let  $N_t^{(6)}$  be the stabilizer of  $\tilde{Z}_{t-1} \cup \{\beta_1, \beta_1^*\}$  in  $W(B_n)$  and  $D_t^{(6)}$  be a set of left coset representatives of  $N_t^{(6)}$  in  $W(B_n)$ . Let  $A_t^{(6)} = \tau A_t^{(2)} \tau^{-1}$  and  $W_t^{(6)} = \tau W_t^{(2)} \tau^{-1}$ .

Finally, for  $i = 1, 2, 3, 4$ , let  $D_{t,L}^{(i)} = D_{t,R}^{(i)} = D_t^{(i)}$ ,  $D_{t,L}^{(5)} = D_{t,R}^{(6)} = D_t^{(5)}$ , and  $D_{t,L}^{(6)} = D_{t,R}^{(5)} = D_t^{(6)}$ .

**Proposition 4.7.** In  $\text{Br}(B_n)$ , the following properties hold for all  $i \in \{1, \dots, 6\}$ .

- (i) For each  $x \in N_t^{(i)}$  we have  $x f_t^{(i)} = f_t^{(i)} x$ .
- (ii) For each  $a \in A_t^{(i)}$ , we have  $a f_t^{(i)} = f_t^{(i)}$ .
- (iii) For each  $b \in W_t^{(i)}$  there exists some  $c \in C_t^{(i)}$ , such that  $b f_t^{(i)} = c f_t^{(i)}$ .
- (iv) For each  $c \in C_t^{(i)}$  we have  $c f_t^{(i)} = c f_t^{(i)}$ .

As a result, each monomial in  $M$  can be written in the normal form

$$u f_t^{(i)} v w,$$

for some  $u \in D_{t,L}^{(i)}$ ,  $w \in (D_{t,R}^{(i)})^{\text{op}}$ ,  $v \in C_t^{(i)}$ , and  $i \in \{1, \dots, 6\}$ .

*Proof.* (i). For  $i \in \{1, 2, 3\}$ , this follows from Definition 3.1 and Lemma 4.5. For  $i \in \{4, 5, 6\}$ , observe that  $N_t^{(i)}$  is the (semi-direct) product of  $W_t^{(i)}$  and  $A_t^{(i)}$ , so (i) will follow from (ii), (iii), and (iv).

(ii). We use the following three equalities.

$$r_i r_{i-1} r_{i+1} r_i e_{i-1} e_{i+1} = e_{i-1} e_{i+1}, \quad \text{for } i > 1 \quad (4.1)$$

$$r_0 e_i e_i^* = e_i e_i^*, \quad (4.2)$$

$$r_i e_i = e_i. \quad (4.3)$$

The first holds as

$$\begin{aligned} r_i r_{i-1} r_{i+1} r_i (e_{i-1} e_{i+1}) &\stackrel{(2.7)}{=} r_i r_{i-1} (r_{i+1} r_i e_{i+1}) e_{i-1} \\ &\stackrel{(2.9)}{=} (r_i r_{i-1} e_i) (e_{i+1} e_{i-1}) \\ &\stackrel{(2.9)}{=} (e_{i-1} e_i e_{i-1}) e_{i+1} \\ &\stackrel{(2.37)}{=} e_{i-1} e_{i+1}, \end{aligned}$$

the second equality is from Lemma 2.8, and the third equality follows from definition. Now for the proof that the lemma holds for each generator, the formulas (4.1)–(4.3) or their conjugations can cover all possible cases.

(iii). The difference of generators of  $W_t$  and  $C_t$  is made up of  $\{r_{n+1-2i}^*\}_{i=2}^t$ , which are conjugate to  $r_{n-1}^*$  by some elements in  $A_t$ , for example

$$r_{n-2} r_{n-3} r_{n-1} r_{n-2} \beta_{n-3}^* = \beta_{n-1}^*.$$

Therefore (iii) for  $f_t^{(1)}$  follows from (ii). We can derive it for  $f_t^{(2)}$  and  $f_t^{(3)}$  by applying Lemma 2.8.

For  $i = 4$ , recall that  $g = e_2 r_1 e_0 e_1 e_2$ . By use of Lemma 2.9 and

$$r_{\epsilon_{n+4-2t}-\epsilon_4} r_2 r_{n+3-2t} r_{\epsilon_{n+4-2t}-\epsilon_4} e_2 e_{n+3-2t} \stackrel{(4.1)}{=} e_2 e_{n+3-2t},$$

the (ii) and (iii) about  $f_t^{(4)}$  can be obtained.

For  $i = 5$ , the (ii) holds in view of  $r_1 r_0 r_1 e_0 = e_0$  and the argument of (ii) for  $f_t^{(3)}$ ; the (iii) holds because of

$$r_2 r_1 r_0 r_1 r_2 e_0 \stackrel{(2.6)}{=} r_2 (r_1 r_0 r_1 e_0) r_2 \stackrel{(2.12)}{=} r_2 e_0 r_2 \stackrel{(2.6)}{=} e_0$$

and (iii) for  $f_t^{(3)}$ .

When  $i = 6$ , (ii) and (iii) hold naturally for (ii) and (iii) of  $f_t^{(2)}$ .

As for the final statement, Note that, if  $w \in W$ , we can write  $w = vn$  with  $v \in D_t^{(i)}$  and  $n \in N_t^{(i)}$ ; but  $n = ba$  with  $b \in W_t^{(i)}$  and  $a \in A_t^{(i)}$ ; so, by (iii),  $w f_t^{(i)} = v b f_t^{(i)} = v f_t^{(i)} c$  for some  $c \in C_t^{(i)}$ . Using the opposition involution of

Proposition 2.5, we can finish for  $i = 1, 2, 3, 4$  as  $(f_t^{(i)})^{\text{op}} = f_t^{(i)}$ . The other two cases can be treated similarly, so we restrict ourselves to  $i = 5$ .

Applying the results obtained so far to image under opposition of  $f_t^{(5)}z$  for  $z \in W$ , we find  $u \in D_t^{(6)} = D_{t,R}^{(5)}$  and  $d \in C_t^{(6)}$  such that  $(f_t^{(5)}z)^{\text{op}} = z^{-1}f_t^{(6)} = u^{-1}f_t^{(6)}d^{-1}$ , so  $f_t^{(5)}z = df_t^{(5)}u$ . As  $d \in C_t^{(6)} = C_t^{(5)}$  (see Definition 4.3), this expression blends well with  $vf_t^{(5)}c$  for  $wf_t^{(5)}$  to give the required normal form for  $wf_t^{(5)}z$ .

(iv). This follows from a straightforward check for each the generators of  $C_t^{(i)}$ .  $\square$

We now come to the complete normal forms result by replacing  $M$  in the last statement of Proposition 4.7 with  $\text{BrM}(\mathbf{B}_n)$ .

**Theorem 4.8.** *Up to powers of  $\delta$ , each monomial in  $\text{BrM}(\mathbf{B}_n)$  can be written in the normal form*

$$uf_t^{(i)}vw,$$

for some  $u \in D_{t,L}^{(i)}$ ,  $w \in (D_{t,R}^{(i)})^{\text{op}}$ ,  $v \in C_t^{(i)}$ , and  $f_t^{(i)} \in F$ .

In view of Proposition 4.7 and Lemma 4.2, for the proof of the theorem it remains to consider the products of the form  $e_\beta f_t^{(i)}$ . This is done in Lemmas 4.9—4.16.

**Lemma 4.9.** *For  $f_t^{(1)}$  and  $f_t^{(2)}$ , the following statements hold.*

(i) *If  $i < n - 2t$ , then there exist  $r, s \in W(\mathbf{B}_n)$  such that*

$$e_{\epsilon_{i+2}}f_t^{(1)} = \delta^t r f_{t+1}^{(3)} r^{-1} \quad \text{and} \quad e_{\epsilon_{i+2}}f_t^{(2)} = s f_{t+1}^{(3)} s^{-1}.$$

(ii) *If  $i \geq n - 2t$ , then there exists  $s \in W(\mathbf{B}_n)$  such that*

$$e_{\epsilon_{i+2}}f_t^{(1)} = \delta^{t-1} s f_t^{(5)} s^{-1} \quad \text{and} \quad e_{\epsilon_{i+2}}f_t^{(2)} = s f_t^{(5)} s^{-1}.$$

As a consequence, for each short root  $\beta$  of  $\Psi^+$  and each  $i \in \{1, 2\}$ , the monomial  $e_\beta f_t^{(i)}$  belongs to  $M$ .

*Proof.* (i). We have  $e_{\epsilon_{i+2}}f_t^{(1)} = e_B = \delta^{-t}e_{\bar{B}}$ , where  $B = \{\epsilon_{i+2}\} \cup Z_t$  and  $\bar{B} = \{\epsilon_{i+2}\} \cup \tilde{Z}_t$  is on the  $W(\mathbf{B}_n)$ -orbit of  $\bar{Z}_{t+1}$ , hence the first equality. The second equality holds by a similar argument.

(ii). First consider the case  $i = n - 2t$ . For  $s = r_{\epsilon_{i+2}-\epsilon_3} r_i r_{i+1} r_{\epsilon_{i+2}-\epsilon_3}$ , we have

$$s\epsilon_{i+2} = \beta_0, \quad s\beta_{n+1-2t} = \beta_1, \quad s\beta_{n+1-2t}^* = \beta_1^*,$$

and so

$$\begin{aligned} s(Z_t \setminus \{\beta_{n+1-2t}\}) &= (Z_t \setminus \{\beta_{n+1-2t}\}), \\ s(\tilde{Z}_t \setminus \{\beta_{n+1-2t}, \beta_{n+1-2t}^*\}) &= (\tilde{Z}_t \setminus \{\beta_{n+1-2t}, \beta_{n+1-2t}^*\}). \end{aligned}$$

Therefore

$$se_{\epsilon_{i+2}}f_t^{(1)}s^{-1} = se_{\epsilon_{i+2}}s^{-1}se_{\beta_{n+1-2t}}s^{-1}sf_{t-1}^{(1)}s^{-1} = e_0e_1f_{t-1}^{(1)} = \delta^{t-1}f_t^{(5)},$$

and similarly for  $e_{\epsilon_{i+2}}f_t^{(2)}$  instead of  $e_{\epsilon_{i+2}}f_t^{(1)}$ .

Next, consider  $i > n - 2t$ . Now  $\beta_{i+1} \in Z_t \subset \tilde{Z}_t$ , and  $r_i r_{i-1} r_{i+1} r_i$  interchanges  $\beta_{i+1}$  and  $\beta_{i-1}$  as well as  $\beta_{i+1}^*$  and  $\beta_{i-1}^*$ ; moreover, it fixes all other elements of  $Z_t$  and  $\tilde{Z}_t$ . Hence  $r_i r_{i-1} r_{i+1} r_i \epsilon_{i+2} = \epsilon_i$  and the lemma holds by induction on  $i$ . For  $\beta_i \in Z_t \subset \tilde{Z}_t$ , note that  $r_i \{\epsilon_{i+2}\} = \{\epsilon_{i+1}\}$  and that  $r_i$  keeps  $Z_t$  and  $\tilde{Z}_t$  invariant. It follows that, by conjugation with  $r_i r_{i-1} r_{i+1} r_i$  and  $r_i$ , the two equalities are brought back to the cases for  $i - 2$  and  $i - 1$ , respectively.

As for the final statement, note that each short root is of the form  $\epsilon_j$  for some  $j \in \{2, \dots, n+1\}$ .  $\square$

Let  $W = W(B_n)$ .

**Lemma 4.10.** *If  $\beta$  is a long root in  $\Psi^+$ , then*

$$\begin{aligned} e_\beta f_t^{(1)} &\in \delta^{\mathbb{Z}} W f_t^{(1)} \cup \delta^{\mathbb{Z}} W f_{t+1}^{(1)} W \cup \delta^{\mathbb{Z}} f_t^{(2)}, \\ e_\beta f_t^{(2)} &\in \delta^{\mathbb{Z}} W f_t^{(2)} \cup \delta^{\mathbb{Z}} W f_{t+1}^{(2)} W. \end{aligned}$$

*Proof.* Let's consider  $e_\beta f_t^{(1)}$ . First, if  $\beta \in Z_t$ , then  $e_\beta f_t^{(1)} = \delta f_t^{(1)}$ . Second, if  $\beta$  is an orthogonal mate of any element in  $Z_t$ , we have

$$e_\beta f_t^{(1)} = e_{\{\beta\} \cup Z_t} = \delta^{t-1} e_{\overline{\{\beta\} \cup Z_t}} = \delta^{t-1} f_t^{(2)},$$

Third, if  $\beta$  and  $\beta' \in Z_t \cup \tilde{Z}_t$  are two long roots and not orthogonal to each other. Then  $e_\beta e_{\beta'} \stackrel{(2.9)}{=} r_{\beta'} r_\beta e_{\beta'}$ , therefore  $e_\beta f_t^{(1)} \in \delta^{\mathbb{Z}} W f_t^{(1)}$ . Fourth,  $\beta$  is not in the above three cases, hence  $\{\beta\} \cup Z_t$  will be on the  $W$ -orbit of  $Z_{t+1}$ , therefore we have  $e_\beta f_t^{(1)} \in \delta^{\mathbb{Z}} W f_{t+1}^{(1)} W$ .

The second claim of  $e_\beta f_t^{(2)}$  holds by a similar argument.  $\square$

**Lemma 4.11.** *For each long root  $\beta \in \Psi^+$ , we have  $e_\beta f_t^{(3)} \in M$ .*



*Proof.* If  $\beta$  is not orthogonal to all roots of  $\tilde{Z}_{t-1}$ , then  $e_\beta f_t^{(3)} = e_\beta f_{t-1}^{(2)} e_0$  by applying Lemma 4.10 and Lemma 4.2.

If  $\beta$  is orthogonal to  $\tilde{Z}_{t-1}$  and  $\beta_0$ , there exists a  $r \in N_t^{(3)}$ , which does not move any element in  $\tilde{Z}_{t-1}$  and  $\beta_0$  and  $r\beta = \beta_{n+1-2t}$ ; hence  $e_\beta f_t^{(3)} = r f_{t+1}^{(3)} r^{-1}$ .

If  $\beta$  is orthogonal to  $\tilde{Z}_{t-1}$  and not orthogonal to  $\beta_0$ , then we can find an element  $r \in N_t^{(3)}$  such that  $r e_1 r^{-1} = e_\beta$ , which implies that  $e_\beta f_t^{(3)} = r^{-1} e_1 e_0 f_{t-1}^{(2)} r \in W f_t^{(6)} W$ , therefore our lemma holds.  $\square$

**Lemma 4.12.** *For each short root  $\beta \in \Psi^+$ , we have  $e_\beta f_t^{(3)}, e_\beta f_t^{(5)} \in M$ .*

*Proof.* There is an index  $i$  such that  $\beta = r_i r_{i-1} \cdots r_1 \beta_0$ . Now

$$\begin{aligned} e_\beta e_0 &= r_i r_{i-1} \cdots r_1 e_0 r_1 (r_2 \cdots r_i e_0) \stackrel{(2.6)}{=} r_i r_{i-1} \cdots r_1 (e_0 r_1 e_0) r_2 \cdots r_i \\ &\stackrel{(2.15)}{=} \delta r_i r_{i-1} \cdots r_1 e_0 r_2 \cdots r_i \stackrel{(2.6)}{=} \delta r_i r_{i-1} \cdots r_1 r_2 \cdots r_i e_0 \\ &\in \delta W e_0, \end{aligned}$$

and the lemma follows as both  $f_t^{(3)}$  and  $f_t^{(5)}$  begin with  $e_0$ .  $\square$

**Lemma 4.13.** *Let  $\beta \in \Psi^+$  be a long root.*

- (i) *If  $\beta$  is not orthogonal to  $\tilde{Z}_{t-1}$ , then  $e_\beta f_t^{(5)} \in \delta^{\mathbb{Z}} W f_t^{(5)}$ .*
- (ii) *If  $\beta = \epsilon_j \pm \epsilon_i$ , with  $2 < i < j < n + 4 - 2t$ , then  $e_\beta f_t^{(5)} \in \delta^{\mathbb{Z}} W f_{t+1}^{(5)} W$ .*
- (iii) *If  $\beta = \epsilon_j \pm \epsilon_3$ , with  $3 < j < n + 4 - 2t$ , then  $e_\beta f_t^{(5)} \in \delta^{\mathbb{Z}} W f_{t+1}^{(3)} W$ .*
- (iv) *If  $\beta = \epsilon_j \pm \epsilon_2$ , with  $2 < j < n + 4 - 2t$ , then  $e_\beta f_t^{(5)} \in \delta^{\mathbb{Z}} W f_{t+1}^{(5)} W$ .*
- (v) *If  $\beta = \beta_1$ , then  $e_\beta f_t^{(5)} = e_1 f_t^{(5)} = r f_t^{(2)} r^{-1}$ , for some  $r \in W$ .*

Thus for each  $\beta \in \Psi^+$  the monomial  $e_\beta f_t^{(5)}$  belongs to  $M$ .

*Proof.* As  $r_{\epsilon_i}(\epsilon_j + \epsilon_i) = \epsilon_j - \epsilon_i$  and  $\{r_{\epsilon_i}\}_{i=2}^{n+1} \subset N_t^{(6)}$ , we only need consider  $\beta = \epsilon_j - \epsilon_i$ . Case (i) can be checked easily. Case (v) holds as

$$e_1 f_t^{(5)} = e_1 e_0 e_1 f_{t-1}^{(2)} \stackrel{(2.13)}{=} e_1 e_1^* f_{t-1}^{(2)} = e_{\{\beta_1, \beta_1^*\} \cup \tilde{Z}_{t-1}},$$

where  $\{\beta_1, \beta_1^*\} \cup \tilde{Z}_{t-1}$  are on the  $W$ -orbits of  $\tilde{Z}_t$ .

After conjugation by suitable elements of  $N_t^{(5)}$ , we can restrict ourselves to  $\beta = \beta_{n+1-2t}, \beta_2, \beta_1 + \beta_2$  for cases (ii), (iii), and (iv), respectively. Case (ii) can be proved easily. Case (iii) holds as  $e_2 f_t^{(5)} = e_0 e_2 e_1 f_{t-1}^{(2)} = e_0 e_2 r_1 r_2 f_{t-1}^{(2)} = e_0 e_2 f_{t-1}^{(2)} r_1 r_2$ , and  $e_0 e_2 f_{t-1}^{(2)}$  is conjugate to  $e_0 f_t^{(2)}$  by some element of  $W(B_n)$ .

Case (iv) holds as  $e_{\beta_1 + \beta_2} e_0 e_1 = r_1 e_2 r_1 e_0 e_1 \stackrel{(2.36)}{=} r_1 e_2 r_1 e_0 e_1 e_2 r_1 r_2 = r_1 g r_1 r_2$ .  $\square$

**Lemma 4.14.** *For each short root  $\beta \in \Psi^+$ , we have  $e_\beta f_t^{(6)} \in M$ .*

*Proof.* If  $\beta$  is equal to  $\beta_0$  or  $\epsilon_3$ , the lemma holds as

$$\begin{aligned} e_0 f_t^{(6)} &= e_0 e_1 e_0 f_{t-1}^{(2)} \stackrel{(2.16)}{=} \delta e_0 f_{t-1}^{(2)} = \delta f_t^{(3)}, \\ e_{\epsilon_3} f_t^{(6)} &= r_1 e_0 r_1 e_1 e_0 f_{t-1}^{(2)} = \delta r_1 e_0 f_{t-1}^{(2)} = \delta r_1 f_t^{(3)}. \end{aligned}$$

If  $\beta = \epsilon_{i+2}$  for  $1 < i < n$ , then  $e_\beta = r_i \cdots r_2 r_1 e_0 r_1 r_2 \cdots r_i$ . Hence

$$\begin{aligned} e_\beta f_t^{(6)} &= r_i \cdots r_2 r_1 e_0 r_1 r_2 \cdots r_i f_t^{(6)} \\ &\stackrel{(2.6)}{=} r_i \cdots r_2 r_1 (e_0 r_1 r_2 e_1 e_0) r_3 \cdots r_i f_{t-1}^{(2)} \\ &\stackrel{(2.9)}{=} r_i \cdots r_2 r_1 (e_0 e_2 e_1 e_0) r_3 \cdots r_i f_{t-1}^{(2)} \\ &\stackrel{(2.7)+(2.16)}{=} \delta r_i \cdots r_2 r_1 e_0 e_2 r_3 \cdots r_i f_{t-1}^{(2)} \\ &= \delta r_i \cdots r_2 r_1 e_0 r_3 \cdots r_i (r_3 \cdots r_i)^{-1} e_2 r_3 \cdots r_i f_{t-1}^{(2)} \\ &\stackrel{(2.6)}{=} \delta r_i \cdots r_2 r_1 r_3 \cdots r_i e_0 (r_3 \cdots r_i)^{-1} e_2 r_3 \cdots r_i f_{t-1}^{(2)}, \\ &\in W e_0 e_{\epsilon_{i+2}-\epsilon_3} f_{t-1}^{(2)}, \\ &\subseteq W e_0 W (f_t^{(2)} \cup f_{t-1}^{(2)}) W \quad \text{by Lemma 4.10} \\ &\subseteq W (f_t^{(3)} \cup f_{t+1}^{(3)} \cup f_t^{(5)} \cup f_{t+1}^{(5)}) W \quad \text{by Lemma 4.9} \\ &\subseteq M. \end{aligned}$$

□

**Lemma 4.15.** *If  $\beta \in \Psi^+$  is a long root, then  $e_\beta f_t^{(6)}$  belongs to  $M$ .*

*Proof.* If  $\beta = \epsilon_{j+2} \pm \epsilon_{i+2}$  with  $1 < i < j < n$ , then the lemma holds as

$$\begin{aligned} e_\beta e_1 e_0 f_{t-1}^{(2)} &= e_1 (e_0 (e_\beta f_{t-1}^{(2)})) \\ &\in e_1 e_0 (\delta^\mathbb{Z} W f_{t-1}^{(2)} W \cup \delta^\mathbb{Z} W f_t^{(2)} W) \quad \text{by Lemma 4.10} \\ &\subseteq \delta^\mathbb{Z} (e_1 W f_t^{(3)} W \cup W f_{t-1}^{(5)} W \cup W f_{t+1}^{(3)} W \cup W f_t^{(5)} W) \quad \text{by Lemma 4.9} \\ &\subseteq M \quad \text{by Lemma 4.11 and Lemma 4.13.} \end{aligned}$$

Otherwise,  $\beta$  is not orthogonal to  $\beta_1$ , and so  $e_\beta e_1 e_0 f_{t-1}^{(2)} \stackrel{(2.9)}{=} r_1 r_\beta e_1 e_0 f_{t-1}^{(2)}$  if  $\beta \notin \{\beta_1, \beta_1^*\}$ , or  $\delta e_1 e_0 f_{t-1}^{(2)}$  if  $\beta \in \{\beta_1, \beta_1^*\}$ . Therefore, the lemma holds. □

Finally, we deal with  $f_t^{(i)}$  for  $i = 4$ .

**Lemma 4.16.** *For each  $\beta \in \Psi^+$ , the monomial  $e_\beta f_t^{(4)}$  belongs to  $M$ .*

*Proof.* First assume that  $\beta$  is a short root. If  $\beta = \epsilon_{i+2}$  with  $i > 2$ , then

$$\begin{aligned}
e_\beta f_t^{(4)} &= e_\beta g f_{t-1}^{(2)} = r_i \cdots r_4 r_3 r_2 r_1 e_0 r_1 r_2 r_3 \cdots r_i g f_{t-1}^{(2)} \\
&\stackrel{(2.6)}{=} r_i \cdots r_4 r_3 r_2 r_1 (e_0 r_1 r_2 r_3 g) r_4 \cdots r_i f_{t-1}^{(2)} \\
&\stackrel{(2.48)}{=} \delta r_i \cdots r_4 r_3 r_2 r_1 (e_0 (e_1 (e_3 r_2 \cdots r_i f_{t-1}^{(2)}))) \\
&\in \delta^{\mathbb{Z}} W e_0 f_{t'}^{(2)} W \quad \text{by Lemma 4.10} \\
&\subseteq M \quad \text{by Lemma 4.9}
\end{aligned}$$

If  $\beta = \epsilon_4, \epsilon_3$ , or  $\beta_0$ , the same argument as above can be applied with (2.49) and (2.46) in Lemma 2.9.

Next assume  $\beta$  is a long root. If  $\beta = \epsilon_{j+2} \pm \epsilon_{i+2}$  with  $2 < i < j < n$ , then  $e_\beta g f_{t-1}^{(2)} = g e_\beta f_{t-1}^{(2)}$ . If  $\beta$  is not orthogonal to  $\tilde{Z}_{t-1}$  (see Proposition 3.3), then either  $e_\beta f_{t-1}^{(2)} = r_s r_\beta f_{t-1}^{(2)}$ , for some  $\beta_s \in \tilde{Z}_{t-1}$  not orthogonal to  $\beta$ , or  $e_\beta f_{t-1}^{(2)} = \delta f_{t-1}^{(2)}$ . Now the lemma holds as  $r_s r_\beta g = g r_s r_\beta$ . Otherwise,  $j < n+2-2t$ , and we can find some element  $r \in \tilde{N}_{t-1}^{(2)}$  such that  $r\beta = \beta_{n+1-2t}$ . Then  $e_\beta f_{t-1}^{(2)} = r^{-1} e_{n+1-2t} r f_{t-1}^{(2)} = r^{-1} e_{n+1-2t} f_{t-1}^{(2)} r = \delta^{-1} r^{-1} f_t^{(2)} r$ , for  $r^{-1}$  commutes with  $g$ , hence the lemma holds by Lemma 4.2.

If  $\beta = \epsilon_{j+2} \pm \epsilon_{i+2}$  with  $0 < i \leq 2 \leq j < n$ , the root  $\beta$  is not orthogonal to  $\beta_2$  or  $\beta \in \{\beta_2, \beta_2^*\}$ , hence  $e_\beta g = r_2 r_\beta g$  or  $\delta g$ , which implies that the lemma holds by Lemma 4.2.

If  $\beta = \epsilon_{j+2} \pm \epsilon_2$ , then  $r_0 \beta = \beta^*$  and  $r_0 \in N_t^{(2)}$ , so it suffices to consider  $\beta = \epsilon_{j+2} - \epsilon_2$ . If  $2 < j$ , then

$$\begin{aligned}
e_\beta f_t^{(2)} &= e_\beta g f_{t-1}^{(2)} = r_j \cdots r_4 r_3 r_2 e_1 r_2 r_3 \cdots r_j g f_{t-1}^{(2)} \\
&= r_j \cdots r_4 r_3 r_2 (e_1 r_2 r_3 g) r_4 \cdots r_j f_{t-1}^{(2)} \\
&\stackrel{(2.50)}{=} r_j \cdots r_4 r_3 r_2 (e_1 (e_0 r_1 r_2 r_3 (e_2 r_4 \cdots r_j f_{t-1}^{(2)}))) \\
&\in \delta^{\mathbb{Z}} W e_1 e_0 W f_{t-1}^{(2)} W \quad \text{by Lemma 4.10} \\
&\subseteq \delta^{\mathbb{Z}} W e_1 W f_{t-1}^{(2)} W \quad \text{by Lemma 4.9} \\
&\subseteq \delta^{\mathbb{Z}} W f_{t-1}^{(2)} W, \quad \text{by Lemma 4.15}
\end{aligned}$$

and we are done. The remaining two cases are  $\beta = \beta_1$  and  $\beta = \beta_1 + \beta_2$ . These follow readily from  $e_1 e_2 = r_2 r_1 e_2$  and  $r_2 e_1 r_2 e_2 = r_1 e_2$ . This proves the lemma.  $\square$

By means of Lemmas 4.9—4.16, we have shown that  $e_\beta f_t^{(i)} \in M$  for each  $\beta \in \Psi^+$  and each  $i \in \{1, \dots, 6\}$ , which suffices to complete the proof of Theorem 4.8.

We proceed to give an upper bound for the rank of the Brauer algebra  $\text{Br}(\mathbb{B}_n)$ . By Theorem 4.8, the upper bound is given by

$$\sum_{i=1}^6 |D_{t,L}^{(i)}| |D_{t,R}^{(i)}| |C_t^{(i)}|. \quad (4.4)$$

Table 1 lists the cardinalities of  $D_{t,L}^{(i)}$ ,  $D_{t,R}^{(i)}$ , and  $C_t^{(i)}$ .

Table 1: Cardinalities of coset and centralizers

$D_0^{(1)}$	1	$C_0^{(1)}$	$2^n n!$
$D_t^{(1)}$	$2^t \binom{n}{2t} t!!$	$C_t^{(1)}$	$2^{n+1-2t} (n-2t)!$
$D_t^{(2)}$	$\binom{n}{2t} t!!$	$C_t^{(2)}$	$(n-2t)!$
$D_t^{(3)}$	$n \binom{n-1}{2t-2} (t-1)!!$	$C_t^{(3)}$	$(n+1-2t)!$
$D_t^{(4)}$	$n \binom{n-1}{2t} t!!$	$C_t^{(4)}$	$(n-1-2t)!$
$D_t^{(5)}$	$n(n-1) \binom{n-2}{2t-2} (t-1)!!$	$C_t^{(5)}$	$(n-2t)!$
$D_t^{(6)}$	$\binom{n}{2t} t!!$	$C_t^{(6)}$	$(n-2t)!$

**Lemma 4.17.** *For  $0 < t \leq (n+1)/2$ ,*

$$\sum_{i=2}^6 |D_{t,L}^{(i)}| |D_{t,R}^{(i)}| |C_t^{(i)}| = \left( \binom{n+1}{2t} t!! \right)^2 (n+1-2t)!.$$

*Proof.* Recall our  $M^{(i)}$ ,  $i = 2, \dots, 6$ , in the proof of Theorem 2.11, and let  $M_t^{(i)}$  be the subset of diagrams  $M^{(i)}$  with  $t$  horizontal strands at the top of their diagrams. These  $M_t^{(i)}$  consist of all possible diagrams with  $t$  horizontal strands in  $T_{n+1}^\pm \cap T_{n+1}^0$ . The count of classical Brauer diagrams (of [2]) related to the Brauer monoid of type  $\text{Br}(\mathbb{A}_n)$  with  $t$  horizontal strands at the top can be conducted as follows. First choose  $2t$  points at the top (bottom) and make  $t$  horizontal strands; the remaining  $n+1-t$  vertical strands correspond to the elements of the Coxeter group of type  $W(\mathbb{A}_{n-t})$ . Therefore the right hand side of the equality is the number of all possible diagrams in  $T_{n+1}^\pm \cap T_{n+1}^0$  with  $t$  horizontal strands at the top, and so equals

$$\sum_{i=2}^6 |M_t^{(i)}| = \left( \binom{n+1}{2t} t!! \right)^2 (n+1-2t)!.$$

We compute  $|M_t^{(i)}|$  for  $i = 2, \dots, 6$ .

For  $i = 2$ , we just count as above with  $n + 1$  replaced by  $n$ , so

$$|M_t^{(2)}| = \left( \binom{n}{2t} t!! \right)^2 (n - 2t)! = |D_t^{(2)}|^2 |C_t^{(2)}| = |D_{t,L}^{(2)}| |D_{t,R}^{(2)}| |C_t^{(2)}|.$$

For  $i = 5$ , we first choose two points from the top  $n + 1$  points except 1 for the horizontal strand from 1 and the vertical strand from  $\hat{1}$ , and we choose  $2(t - 1)$  points at the top from the remaining  $n + 1 - 3$  points at the top for  $t - 1$  horizontal strands and  $2t$  points at the bottom  $n + 1$  points except  $\hat{1}$  for  $t$  horizontal strands, and then the vertical strands between the remaining  $n - 2t$  points at the top and the remaining  $n - 2t$  points at the bottom will be corresponding to elements of the Coxeter group of type  $W(A_{n-2t-1})$ . Therefore,

$$\begin{aligned} |M_t^{(5)}| &= n(n - 1) \binom{n - 2}{2t - 2} (t - 1)!! \binom{n}{2t} t!! (n - 2t)! \\ &= |D_t^{(5)}| |D_t^{(6)}| |C_t^{(5)}| = |D_{t,L}^{(5)}| |D_{t,R}^{(5)}| |C_t^{(5)}|. \end{aligned}$$

By reversing the top and bottom, we obtain a one to one correspondence between  $M_t^{(5)}$  and  $M_t^{(6)}$ ; it follows that

$$|M_t^{(6)}| = |M_t^{(5)}| = |D_{t,L}^{(6)}| |D_{t,R}^{(6)}| |C_t^{(6)}|.$$

For  $i = 4$ , we first choose one point from the bottom (top)  $n + 1$  points except  $\hat{1}$  (1) for the vertical strand from 1 ( $\hat{1}$ ); the remaining count of horizontal strands and other vertical stands is as in the classical case after replacing  $n + 1$  by  $n - 1$ ; hence

$$|M_t^{(4)}| = n^2 \left( \binom{n - 1}{2t} t!! \right)^2 (n - 1 - 2t)! = |D_t^{(4)}|^2 |C_t^{(2)}| = |D_{t,L}^{(4)}| |D_{t,R}^{(4)}| |C_t^{(4)}|.$$

For  $i = 3$ , we first choose one point at the top (bottom) distinct from 1 ( $\hat{1}$ ) for the horizontal strand from 1 ( $\hat{1}$ ); the remaining count of other horizontal strands and vertical stands is as in the classical case after replacing  $n + 1$  by  $n - 1$  and  $t$  by  $t - 1$ ; it follows that

$$\begin{aligned} |M_t^{(3)}| &= n^2 \left( \binom{n - 1}{2t - 2} (t - 1)!! \right)^2 (n + 1 - 2t)! = |D_t^{(3)}|^2 |C_t^{(3)}| \\ &= |D_{t,L}^{(3)}| |D_{t,R}^{(3)}| |C_t^{(3)}|. \end{aligned}$$

The equality of the lemma now follows from the above 5 equalities for  $M_t^{(i)}$ .  $\square$

**Corollary 4.18.** *The algebra  $\text{Br}(\mathbf{B}_n)$  has a spanning set over  $\mathbb{Z}[\delta^{\pm 1}]$  of size at most  $f(n)$ .*

*Proof.* By (4.4), the rank of  $\text{Br}(\mathbf{B}_n)$  is at most

$$\begin{aligned} & |W(\mathbf{B}_n)| + \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} |D_{t,L}^{(1)}| |D_{t,R}^{(1)}| |C_t^{(1)}| + \sum_{t=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \binom{n+1}{2t} t!! \right)^2 (n+1-2t)! \\ &= 2^n \cdot n! + 2^n \sum_{t=1}^{\lfloor \frac{n}{2} \rfloor} \left( \binom{n}{2t} t!! \right)^2 (n-2t)! + \sum_{t=1}^{\lfloor \frac{n+1}{2} \rfloor} \left( \binom{n+1}{2t} t!! \right)^2 (n+1-2t)!. \end{aligned}$$

From [17], it follows that

$$\sum_{t=0}^{\lfloor \frac{k}{2} \rfloor} \left( \binom{k}{2t} t!! \right)^2 (k-2t)! = k!!.$$

By applying this for  $k = n, n+1$  to the last two summands of the above equality, we obtain that the rank of  $\text{Br}(\mathbf{B}_n)$  is at most  $2^{n+1} \cdot n!! - 2^n \cdot n! + (n+1)!! - (n+1)! = f(n)$ .  $\square$

We end this section with a proof of Theorem 1.2. By Corollary 4.18 there is a spanning set of  $\text{Br}(\mathbf{B}_n)$  of size  $f(n)$ . By Theorem 2.11, this set maps onto a spanning set of  $\text{SBr}(\mathbf{D}_{n+1})$  of size at most  $f(n)$ . Moreover, by the same theorem,  $\text{SBr}(\mathbf{D}_{n+1})$  is a free algebra over  $\mathbb{Z}[\delta^{\pm 1}]$  of rank  $f(n)$ . This implies that the spanning set of  $\text{Br}(\mathbf{B}_n)$  is a basis and that  $\text{Br}(\mathbf{B}_n)$  is free of rank  $f(n)$ . In particular,  $\phi : \text{Br}(\mathbf{B}_n) \rightarrow \text{SBr}(\mathbf{D}_{n+1})$  is an isomorphism and Theorem 1.2 is proved.

## 5 Cellularity

Recall from [14] that an associative algebra  $A$  over a commutative ring  $R$  is cellular if there is a quadruple  $(\Lambda, T, C, *)$  satisfying the following three conditions.

(C1)  $\Lambda$  is a finite partially ordered set. Associated to each  $\lambda \in \Lambda$ , there is a finite set  $T(\lambda)$ . Also,  $C$  is an injective map

$$\coprod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \rightarrow A$$

whose image is called a *cellular basis* of  $A$ .

(C2) The map  $*$  :  $A \rightarrow A$  is an  $R$ -linear anti-involution such that  $C(x, y)^* = C(y, x)$  whenever  $x, y \in T(\lambda)$  for some  $\lambda \in \Lambda$ .

(C3) If  $\lambda \in \Lambda$  and  $x, y \in T(\lambda)$ , then, for any element  $a \in A$ ,

$$aC(x, y) \equiv \sum_{u \in T(\lambda)} r_a(u, x)C(u, y) \pmod{A_{<\lambda}},$$

where  $r_a(u, x) \in R$  is independent of  $y$  and where  $A_{<\lambda}$  is the  $R$ -submodule of  $A$  spanned by  $\{C(x', y') \mid x', y' \in T(\mu) \text{ for } \mu < \lambda\}$ .

Such a quadruple  $(\Lambda, T, C, *)$  is called a *cell datum* for  $A$ .

**Theorem 5.1.** *There is a cellular datum for  $\text{Br}(\mathbb{B}_n) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} R$  if  $R$  is an integral domain in which 2 and  $\delta$  are invertible elements.*

*Proof.* Let  $R$  be as indicated and write  $A = \text{Br}(\mathbb{B}_n) \otimes_{\mathbb{Z}[\delta^{\pm 1}]} R$ . We introduce a quadruple  $(\Lambda, T, C, *)$  and prove that it is a cell datum for  $A$ . The map  $*$  on  $A$  will be the natural anti-involution  $\cdot^{\text{op}}$  on  $A$  over  $R$ . By Proposition 2.5, the natural anti-involution is an  $R$ -linear anti-involution of  $A$ .

By Theorem 4.8 and Theorem 1.2, the Brauer algebra  $A$  over  $R$  has a basis consisting of the elements of (i)–(vi) in Theorem 4.8.

For  $t \in \{0, \dots, \lfloor n/2 \rfloor\}$ , let  $C_t^* = \psi\phi(C_t^{(1)})$  and  $C_t = \langle \psi(R_2), \psi\phi(C_t^{(2)}) \rangle \cong W(A_{n-2t}) \subset W(D_{n+1})$ , and put  $Y = C_t^*$  or  $C_t$ . As  $Y$  is a Weyl group with irreducible factors of type B or A and the coefficient ring  $R$  satisfies the conditions of [12, Theorem 1.1], we conclude from [12, Corollary 3.2] that the group ring  $R[Y]$  is a cellular subalgebra of  $A$ . Let  $(\Lambda_Y, T_Y, C_Y, *_Y)$  be the corresponding cell datum for  $R[Y]$ . By [12, Section 3],  $*_Y$  is the map  $\cdot^{\text{op}}$  on  $R[Y]$ .

The underlying set  $\Lambda$  is defined as the union of  $\Lambda_1$  and  $\Lambda_2$ , where  $\Lambda_1 = \{t\}_{t=0}^{\lfloor \frac{n}{2} \rfloor}$ ,  $\Lambda_2 = \{(t, \theta)\}_{t=1}^{\lfloor \frac{n+1}{2} \rfloor}$ . A set of  $t$  pairs in  $\{1, \dots, n+1\}$  is called *admissible  $t$ -set* in  $\{1, \dots, n+1\}$  if no two pairs have a common number. We denote the set of all admissible  $t$ -sets of  $\{1, \dots, n+1\}$  by  $U_t^{n+1}$ . A decorated pair in  $\{1, \dots, n+1\}$  is a triple  $\{i, j, +\}$  or  $\{i, j, -\}$  with  $1 \leq i, j \leq n+1$  with  $\pm$  for decorations. A *decorated admissible  $t$ -set* in  $\{1, \dots, n+1\}$  is some admissible  $t$ -set in  $\{1, \dots, n+1\}$  with each pair being decorated. We denote all decorated admissible  $t$ -sets in  $\{1, \dots, n+1\}$  by  $U_t^{*n+1}$ . The set of all decorated admissible  $t$ -sets in  $\{1, \dots, n+1\}$  without 1 appearing in any pair is denoted by  $U_t^{|*n+1}$ . We view  $U_t^{n+1}$  as the subset of  $U_t^{*n+1}$  of all admissible  $t$ -sets all of whose pairs are decorated by  $-$ . For each  $t \in \Lambda_1$ , we define the associated finite set to be

$$T(t) = \{(u, v) \mid u \in U_t^{|*n+1}, v \in T_{C_t^*}\}.$$

For each  $(t, \theta)$ , we define the associated finite set to be

$$T((t, \theta)) = \{(u, v) \mid u \in U_t^{n+1}, v \in T_{C_t}\}.$$

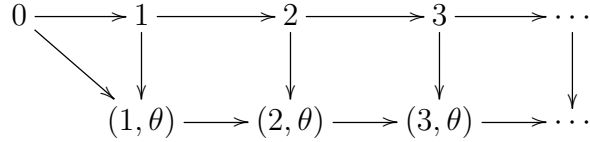
By Theorem 2.11 and (C1) for  $R(Y)$ , there exists a map

$$D : \coprod_{\lambda \in \Lambda} T(\lambda) \times T(\lambda) \rightarrow \text{BrMD}(\text{B}_n),$$

where the diagram of  $D((u_1, v_1), (u_2, v_2))$  is given as the top horizontal strands are strands between pairs in  $u_1$  and decorated for  $+$ , the bottom horizontal strands are similarly given by  $u_2$  the free (decorated) vertical strands and multiplied by  $\xi\delta$  or not are given by  $C_Y(v_1, v_2)$ , multiplied by  $\theta\delta$  if  $\lambda \in \Lambda_2$ . We see  $D$  is an injective map and its image is a basis of  $\text{BrMD}(\text{B}_n)$ . Therefore we define  $C = \phi^{-1}\psi^{-1}D$ . The partial order on  $\Lambda$  is given by

- $\lambda_1 > \lambda_2$  if  $\lambda_1 = t_1 < t_2 = \lambda_2 \in \Lambda_1$ ,
- (2)  $\lambda_1 > \lambda_2$  if  $\lambda_1 = (t_1, \theta)$ ,  $\lambda_2 = (t_2, \theta) \in \Lambda_2$  and  $t_1 < t_2$ ,
- (3)  $\lambda_1 > \lambda_2$  if  $\lambda_1 = t_1 \in \Lambda_1$  and  $\lambda_2 = (t_2, \theta) \in \Lambda_2$  and  $t_1 \leq t_2$ .

It can be illustrated by the following Hasse diagram, where  $a > b$  is equivalent to the existence of a directed path from  $a$  to  $b$ .



In other words, we inherit the cellular structure of  $\text{Br}(\text{D}_{n+1})$  in [4, section 6]. By Theorem 1.2 and Theorem 2.11, the quadruple  $(\Lambda, T, C, *)$  satisfies (C1). From the diagram representation of  $\text{BrMD}(\text{B}_n)$  described in Theorem 2.11 and (C2) of  $R[Y]$ , the quadruple  $(\Lambda, T, C, *)$  satisfies (C2) with  $*$  =  $\cdot^{\text{op}}$ . It remains to check condition (C3) for  $(\Lambda, T, C, *)$ . For this we just need to consider  $r_i C((u_1, v_1), (u_2, v_2))$  and  $e_i C((u_1, v_1), (u_2, v_2))$ . This can be proved by a case-by-case check using the lemmas in Section 4 or by an argument using the diagram representation of  $\text{BrMD}(\text{B}_n)$  and (C3) of  $R[Y]$ .

We conclude that  $(\Lambda, T, C, *)$  is a cell datum for  $\text{Br}(\text{B}_n) \otimes_{\mathbb{Z}[\delta \pm 1]} R$ .  $\square$

*Remark 5.2.* In [15], König and Xi proved that Brauer algebras of type A are inflation cellular algebras, and also in [1], Bowman proved that the Brauer algebras found in [8] cellularly stratified algebras (a stronger version of inflation cellular algebras). Both kinds of algebras have totally ordered sets  $\Lambda$  associated to the cellular structures. But Brauer algebras of type D and type B, just have partially ordered sets  $\Lambda$  in the cell data given above. this explains why we have not been able to use [15] for a cellularity proof.



## References

- [1] C. Bowman, Brauer algebras of type  $C$  are cellular stratified algebras, [arXiv:1102.0438v1](#), [math.RT], 2 Feb 2011.
- [2] R. Brauer, On algebras which are connected with the semisimple continuous groups, *Annals of Mathematics*, **38** (1937), 857–872.
- [3] Zhi Chen, Flat connections and Brauer Type Algebras, [arXiv:1102.4389v1](#), [math.RT], 22 Feb 2011.
- [4] A.M. Cohen, B. Frenk and D.B. Wales, Brauer algebras of simply laced type, *Israel Journal of Mathematics*, **173** (2009) 335–365.
- [5] A.M. Cohen, D.A.H. Gijsbers & D.B. Wales, *BMW algebras of simply laced type*, *J. Algebra*, **286** (2005) 107–153.
- [6] A.M. Cohen, Dié A.H. Gijsbers and D.B. Wales, The BMW Algebras of type  $D_n$ , [arXiv:0704.2743](#), [math.RT], April 2007.
- [7] A.M. Cohen, Dié A.H. Gijsbers and D.B. Wales, Tangle and Brauer diagram algebras of type  $D_n$ , *Journal of Knot Theory and Its Ramifications*, Vol 18, No 4, April 2009, 447–483.
- [8] A.M. Cohen, S. Liu and S. Yu, Brauer algebras of type  $C$ , *Journal of Pure and Applied Algebra*, **216** (2012) 407–426.
- [9] A.M. Cohen, D.B. Wales, *The Birman–Murakami–Wenzl Algebras of Type  $E_n$* , *Transformation Groups*, **16** (2011) 681–715.
- [10] J. Crisp, Injective maps between Artin groups, in *Geometric Group Theory Down Under*, Lamberra 1996 (J. Cossey, C.F. Miller III, W.D. Neumann and M.Shapiro, eds.) De Gruyter, Berlin, 1999, 119–137.
- [11] T. tom Dieck, Quantum groups and knot algebra, Lecture notes, May 4, 2004.
- [12] M. Geck, Hecke algebras of finite type are cellular, *Inventiones Mathematicae*, **169** (2007), 501–517.
- [13] J. J. Graham, Modular representations of Hecke algebras and related algebras, Ph. D. thesis, University of Sydney (1995).
- [14] J.J. Graham and G.I. Lehrer, Cellular algebras, *Inventiones Mathematicae* **123** (1996), 1–44.

- [15] S. König and C. Xi, Cellular algebras: inflations and Morita equivalences, J. London Math. Soc. (2)**60** (1999), 700–722 **123** (1996), 1–44.
- [16] R. Häring-Oldenburg, The reduced Birman-Wenzl algebra of Coxeter type of B, Journal of Algebra **213**, 437–466, 1999.
- [17] M. Petkovsek, H.S. Wilf and D. Zeilberger, A=B, A.K. Peters, Wellesley MA, 1996. <http://www.cis.upenn.edu/~wilf>